Journal of Statistical Physics, Vol. 116, Nos. 5/6, September 2004 (© 2004)

On Isotropic Distributional Solutions to the Boltzmann Equation for Bose-Einstein Particles

Xuguang Lu[1]

Received October 9, 2003; accepted February 9, 2004

The paper considers the spatially homogeneous Boltzmann equation for Bose-Einstein particles (BBE). In order to include the hard sphere model, the equation is studied in a weak form and its solutions (including initial data) are set in the class of isotropic positive Borel measures and therefore called isotropic distributional solutions. Stability of distributional solutions is established in the weak topology, global existence of distributional solutions that conserve the mass and energy is proved by weak convergence of approximate L^1-solutions, and moment production estimates for the distributional solutions are also obtained. As an application of the weak form of the BBE equation, it is shown that a Bose-Einstein distribution plus a Dirac δ-function is an equilibrium solution to the BBE equation in the weak form if and only if it satisfies a low temperature condition and an exact ratio of the Bose-Einstein condensation.

KEY WORDS: Boltzmann equation; Bose-Einstein particles; hard sphere model; distributional solution; equilibrium; Bose-Einstein condensation.

1. INTRODUCTION

The Boltzmann equation for Bose-Einstein particles under consideration is given by

$$\frac{\partial}{\partial t} f(v,t) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\Big[f' f'_*(1+\varepsilon f)(1+\varepsilon f_*) - f f_*(1+\varepsilon f')(1+\varepsilon f'_*)\Big] d\omega\, dv_* \qquad \text{(BBE)}$$

[1]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China; e-mail: xglu@math.tsinghua.edu.cn

which describes time-evolution of dilute, space-homogeneous, and one species Bose gases. Physical background and derivation of the BBE equation can be found in Chapman and Cowling[9] Chap.17 and Uehling and Uhlenbeck[28]. In Eq. (BBE), the solution f is the velocity distribution (or the number density) of particles, ε is a positive quantum parameter: $\varepsilon=(h/m)^3/\mathrm{g}$ where h is the Planck's constant, m is the particle mass and g is the spin degeneracy. Note that since the common term $ff_*f'f'_*$ in the collision integral can be cancelled automatically, the Eq. (BBE) is equivalent to

$$\frac{\partial}{\partial t}f(v,t)=Q_B(f)(v,t),\quad v\in\mathbf{R}^3,\quad t\in[0,\infty),\tag{1.1}$$

$$Q_B(f)(v,t)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)\Big[f'f'_*(1+\varepsilon f+\varepsilon f_*)-ff_*(1+\varepsilon f'+\varepsilon f'_*)\Big]d\omega\,dv_*.\tag{1.2}$$

In this paper, Eq. (BBE) is understood as its equivalent form Eqs. (1.1)–(1.2). Here f_*, f', f'_* stand for the same density f at different velocities: $f=f(v,t)$, $f_*=f(v_*,t)$, $f'=f(v',t)$, $f'_*=f(v'_*,t)$, where v, v_* and v', v'_* are velocities of two particles before and after their collision respectively. The particle collision is assumed to conserve the momentum and the kinetic energy:

$$v'+v'_*=v+v_*,\quad |v'|^2+|v'_*|^2=|v|^2+|v_*|^2$$

which can be written as an explicit relation:

$$v'=v-\langle v-v_*,\omega\rangle\omega,\quad v'_*=v_*+\langle v-v_*,\omega\rangle\omega,\quad \omega\in\mathbf{S}^2.\tag{1.3}$$

The function $B(v-v_*,\omega)$ – often called the collision kernel – depends on the molecular interactions and is a *nonnegative* Borel function of $|v-v_*|$ and $|\langle v-v_*,\omega\rangle|$ only:

$$B(v-v_*,\omega)\equiv B(|v-v_*|,\cos\theta),\quad \theta=\arccos\left(|\langle v-v_*,\omega\rangle|/|v-v_*|\right).\tag{1.4}$$

(For $v=v_*$ we define $\theta=0$.) In this paper, we do not distinguish between $B(v-v_*,\omega)$ and $B(|v-v_*|,\cos\theta)$. For the hard sphere model, B is given by

$$B(|v-v_*|,\cos\theta)=|v-v_*|\cos\theta\tag{1.5}$$

which is the only original physical model that our main results of the present paper can include. For the models of inverse-power law potentials, B takes the form

$$B(|v-v_*|, \cos\theta) = b(\cos\theta)|v-v_*|^{\gamma}, \quad \gamma < 1 \tag{1.6}$$

where γ is the constant whose value determines different models: hard potentials ($\gamma > 0$), the Maxwellian model ($\gamma = 0$) and soft potentials ($\gamma < 0$). Here the angular function $b(\cos\theta)$ is continuous on $\theta \in [0, \pi/2)$ and has a non-integrable singularity at $\theta = \pi/2$:

$$b(\cos\theta) \sim \text{const.}(\pi/2 - \theta)^{-(3-\gamma)/2}, \quad \theta \to \pi/2 \tag{1.7}$$

which is an effect of huge amount of grazing collisions of particles (Cercignani[(6)], Villani[(30)]). It is different from the classical Boltzmann equation (i.e., $\varepsilon = 0$) that this angular singularity can not be absorbed even in an isotropic weak form of Eq. (BBE) (see Proposition 2 below).

In kinetic theory, the BBE model attracts research interests in its amazing quantum effect. At very low temperatures the quantum effect becomes important and yields the Bose-Einstein Condensation (BEC). The *formation* of BEC can be described by the BBE model and its modifications (Semikov and Tkachev[(24,25)]). On the other hand, it is this quantum effect that makes the rigorous study of the BBE model to be the most difficult in almost all main aspects in comparison with those of the already difficult classical Boltzmann model (see Cercignani, Illner and Pulvirenti[(7)], Villani[(30)] for relevant results and reviews) and the Fermi-Dirac model (which is given by replacing ε with $-\varepsilon$, and this implies that solutions are bounded from above by $1/\varepsilon$ provided that their initial data have this bound; basic results have been obtained in for instance Dolbeault[(10)], Lions[(18)], Lu[(21)], Lu and Wennberg[(22)], but the study on long-time behavior are far from complete). So far there have been no results on the global existence of solutions (with general initial data) to Eq. (BBE) for any collision kernel without cutoff. Partial results are referred to for instance Lu[(20)] for global existence and long-time behavior of isotropic L^1-solutions of Eq. (BBE) with a strong cutoff condition on the kernel (see Eq. (1.8) or (2.1) below), to Escobedo, Mischler and Valle[(13)] for (among their many main results) the global existence and uniqueness of measure-valued solutions of Eq. (BBE) with a similar cutoff condition on the kernel, using rigorously Dirac δ-functions. For similar quantum kinetic models we refer to Escobedo and Mischler[(11)] for the global existence and uniqueness of solutions to the Boltzmann-Compton equation (which describes evolution of a photon gas), long-time behavior on the Bose condensates, and a rigorous justification of an approximation to the Boltzmann-Compton equation by the Kompaneets equation, and to Escobed[12] for further results on the two equations. It is also profit to see Villani[30] Chap. 5 for comments and questions about the BBE-like models.

In this paper, we are mainly concerned with the global existence of isotropic solutions of Eq. (BBE) where the collision kernel includes at least a known physical model, for instance, the hard sphere model. All main difficulties in the study of the BBE model concentrate on the stronger nonlinear term $ff'f'_*$ in the collision integral. It is this nonlinear term that yields BEC, but which constrains the possibilities for investigation: Among all the known physical collision models, we could only deal with (at least in the present level) the hard sphere model and even in this case the Eq. (BBE) has to be taken certain *weak* form (whose solutions are then called weak or distributional solutions) as treated for the classical Boltzmann equation (see e.g., Arkeryd[1], Goudon[15], Villani[29]). Weak solutions are usually obtained by convergence of approximate solutions which are solutions of approximate equations where the original kernel has been made cutoffs so as to overcome the divergence. For instance if B is made a cutoff such that

$$B(|v-v_*|,\cos\theta)\leqslant K|v-v_*|^3\cos^2\theta\sin\theta \tag{1.8}$$

with any given constant $0<K<\infty$, then for all isotropic functions $f,g,h\in L^1(\mathbf{R}^3)$,

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v*,\omega)|f(v)g(v')h(v'_*)|d\omega dv dv_*\leqslant 2K\|f\|_{L^1}\|g\|_{L^1}\|h\|_{L^1} \tag{1.9}$$

(see ref. 20). Without cutoff as (1.8), the collision integral will be divergent for some isotropic functions. In fact, if there is constant $0\leqslant p<\infty$ such that

$$B(|v-v_*|,\cos\theta)\geqslant(|v-v_*|\cos\theta)^p\mathbf{1}_{\{|v-v_*|\cos\theta\,\leqslant\,1\}},$$

then for any isotropic function f in $L^1(\mathbf{R}^3)$ satisfying $f(v)\geqslant|v|^{-5/2}\mathbf{1}_{\{|v|\leqslant 1\}}$, we have (using Carleman representation (see Section 2)) that

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v*,\omega)f(v)f(v')f(v'_*)d\omega dv dv_*=\infty.$$

It should be noted that the cutoff condition as Eq. (1.8) is not enough to insure the convergence of the collision integral; the isotropic condition can not be removed from the bound like (1.9). In fact we have the following

Proposition 1. Let $B(V, \tau)$ be a nonnegative Borel function on $\mathbf{R}_+ \times [0, 1]$ satisfying that $B(V, \tau)$ is continuous in the open set $(0, \infty) \times (0, 1)$ and $B(V_0, \tau_0) > 0$ for some $(V_0, \tau_0) \in (0, \infty) \times (0, 1)$. Let $B(v - v_*, \omega)$ be defined by $B(V, \tau)$ through (1.4). Then there exist non-isotropic and nonnegative functions $f_n \in C_c^\infty(\mathbf{R}^3)$ satisfying $\sup_{n \geqslant 1} \|f_n\|_{L^1} < \infty$ such that for every function $\Psi \in C_b(\mathbf{R}^9)$ satisfying $\Psi(0, x, y) \neq 0$ for all $x, y \in \mathbf{R}^3 \setminus \{0\}$ with $x \perp y$, we have, as $n \to \infty$,

$$\iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) |\Psi(v, v', v'_*)| f_n(v) f_n(v') f_n(v'_*) d\omega dv_* dv \to \infty. \tag{1.10}$$

The proof of this proposition will be given in Section 2. Note that if we choose $\Psi(v, v', v'_*) = \psi(v) + \psi(v' + v'_* - v) - \psi(v') - \psi(v'_*)$ with (for instance) $\psi(v) = e^{-c|v|^2}$ $(c > 0)$, then Ψ satisfies the condition in Proposition 1. This means that for non-isotropic initial data it will be very difficult to prove the existence of global solutions of Eq. (BBE) even in a weak form and with a cutoff (1.8). However there has been no counterexample showing that the global existence of solutions with general (non-isotropic) initial data is impossible. In fact Proposition 1 does not exclude the following possibility: The total integral

$$\int_0^t d\tau \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) f(v, \tau) f(v', \tau) f(v'_*, \tau) d\omega dv_* dv$$

could be convergent even though the $d\omega dv_* dv$-integral is unbounded on $\tau \in [0, t]$. This possibility together with the exponential form of Eq. (BBE) (a Duhamel-type formula where all terms are positive) may be helpful to save the game. But the analysis is of course rather complicated.

In this paper we only consider isotropic solutions of Eq. (BBE) in a weak form where the physical model included is the hard sphere model. Moreover as considered in ref. 13, in order to cover the Bose-Einstein condensation (i.e. the velocity concentration), we need to set the initial data and therefore the corresponding solutions in the largest class: the positive Borel measures. This setting is of course reasonable in physics and has been studied in the classical model where the probability idea and certain contraction properties of probability distributions are used to deal with the Maxwellian molecules (i.e., $\gamma=0$) and the results show that the regularity of solutions is not worse than their initial data in the sense that if an initial datum is in $L^1(\mathbf{R}^3)$ then so is the corresponding solution for all time, and for every initial datum (whatever it is in L^1 or just a singular measure) the solution always converges to a Maxwellian distribution as time goes to infinity unless the initial datum and therefore the solution are the same Dirac measure (see e.g., Gabetta, Toscani and Wennberg,[14] Toscani and Villani,[26] Carlen, Carvalho and Gabetta,[4] Carlen and Lu[5]). But for the quantum model, this measure setting seems to be the only choice at least in the present level because it is far from clear whether a solution of Eq. (BBE) with an initial datum in $L^1(\mathbf{R}^3)$ can be still in $L^1(\mathbf{R}^3)$ for a finite or infinite time interval. This problem is of course closely related to the physical question that for the BBE model and for L^1-initial data whether the BEC can happen within a finite time.

In general, isotropic solutions are functions of $(|v-v_0|, t)$ where v_0 is the mean velocity. Since after a velocity translation the function $f(v+v_0, t)$ is still a solution with initial datum $f_0(v+v_0)$, we may assume that $v_0=0$, so the isotropic solution under consideration takes the form $f=f(v,t)=f(r,t)$ with $r=|v|$.

As mentioned above, in our main results, a weak solution to Eq. (BBE) will be a family of positive Borel measures $\{F_t\}_{t\geqslant 0}$ on $\mathbf{R}_+:=[0,\infty)$, which is a weak limit of approximate L^1-isotropic solutions $\{f^n(r,t)\}$ obtained in ref. 20. Therefore the weak convergence should be taken in the corresponding weak topology. This needs the collision kernel having no singularity.

In this paper we assume that the kernel B satisfies the following conditions (i)–(ii):

$$\text{(i)}\qquad B(\cdot,\cdot)\in C(\mathbf{R}_+\times[0,1]),\qquad \int_0^1 B(V,\tau)d\tau>0\quad \forall\, V>0.$$

$$\text{(ii)}\qquad \sup_{V\geqslant 0,\,\tau\in[0,1]}\frac{B(V,\tau)}{1+V}<\infty,\quad \sup_{V\geqslant 0}\frac{B(V,\tau)}{1+V}\to 0\quad \text{as}\quad \tau\to 0+.$$

The limit in the condition (ii) implies that $B(V,0)=0$ which is necessary for the continuity of some induced kernel in the collision integral of $ff'f'_*$ (see Proposition 2 in Section 2). It is obvious that the conditions (i)–(ii) include the hard sphere model (1.5): $B(V,\tau)=V\tau$. For hard potentials and the Maxwellian model (1.6) ($0\leqslant\gamma<1$), the conditions (i)–(ii) are simply equivalent to the cutoff conditions that the angular function $b(\tau)$ is continuous on $(0,1]$ with $\int_0^1 b(\tau)d\tau>0$ and $b(0):=b(0+)=0$. Of course the condition $b(0+)=0$ is very restrictive in contrast to the severe singularity (1.7). We do not know whether the condition $b(0+)=0$ (or generally $B(V,0+)=0$) is essential, but in our framework it is almost compulsory for proving the weak convergence of approximate solutions.

Our main results of this paper can be summarized as follows (the precise statements will be given in the following sections):

- *Weak Stability*: Let B satisfy (i)–(ii), B_n be given by the cutoff (2.1) ($\forall n \geqslant 1$) or $B_n = B$ ($\forall n \geqslant 1$). Let $F_0^n \geqslant 0$ be a sequence of initial data with bounded mass and energy, and let $F_t^n \geqslant 0$ be conservative isotropic distributional solutions to Eq. (BBE) in the weak form (2.12) with kernels B_n and $F_t^n|_{t=0} = F_0^n$. If F_0^n converges to F_0 weakly in the measure space, then there exist a subsequence $F_t^{n_j}$ and a solution F_t of Eq. (BBE) in the weak form, such that $F_t|_{t=0} = F_0$ and $F_t^{n_j}$ converges weakly to F_t $(j \to \infty)$ for all $t \geqslant 0$. Moreover if $\int_{\mathbf{R}_+} r^2 \, dF_0^n(r) \to \int_{\mathbf{R}^3} r^2 \, dF_0(r)$ $(n \to \infty)$, then F_t also conserves the energy.

- *Existence and Moment Estimates*: Under the conditions (i)–(ii) and for any isotropic initial datum $F_0 \geqslant 0$ having finite mass and energy, the Eq. (BBE) in the weak form (2.12) has an isotropic distributional solution $F_t \geqslant 0$ satisfying $F_t|_{t=0} = F_0$, and F_t conserves the mass and energy. Moreover for hard potentials $(0 < \gamma \leqslant 1)$ satisfying the conditions (i)–(ii), the solution F_t can be chosen such that the moment production estimate (5.4) holds for all high order moments.

- *Equilibrium States and BEC*: The Bose-Einstein distributions and Dirac δ-functions are all equilibrium solutions to Eq. (BBE) in the weak form (2.12). While the sum of a Bose-Einstein distribution and a Dirac δ-function is an equilibrium solution to Eq. (BBE) in the weak form if and only if it satisfies a low temperature condition $T \leqslant T_c$ and an exact ratio Bose-Einstein condensation: $N_0/N = 1 - (T/T_c)^{3/2}$ (which is essentially the same to the traditional result for ideal Bose gases).

The rest of the paper is organized as follows: In Section 2, we first derive a weak form of Eq. (BBE) for approximate solutions for cutoff kernels B_n and then naturally introduce the Definition of weak (or distributional) solutions for the kernel B satisfying the conditions (i)–(ii). We also show in that section the necessity of $B(V, 0) = 0$ and prove the divergence (1.10). In Section 3, we prove some lemmas about continuity and convergence properties which will be used in dealing with weak convergence of (approximate) solutions. Then in Section 4 we prove the weak stability and existence of distributional solutions. In Section 5 we prove the property of non-decrease of energy for all solutions. This implies that the solution obtained in Section 4 conserves the energy. And as an application of the weak stability we also establish in Section 5 the moment production estimates which will be used for the further study of Eq. (BBE). Finally, Section 6 is a simple but important application of the weak form of Eq. (BBE) where we prove the above mentioned results on equilibrium states and BEC.

2. FROM L^1-SOLUTIONS TO DISTRIBUTIONAL SOLUTIONS

Introduce weighted L^1-function spaces

$$L^1_s(\mathbf{R}^3) = \left\{ f \in L^1(\mathbf{R}^3) \,\middle|\, \|f\|_{L^1_s} := \int_{\mathbf{R}^3} (1+|v|^s)|f(v)|dv < \infty \right\}, \quad s \geqslant 0.$$

For each $n \geqslant 1$, consider the following cutoff:

$$B_n(|v-v_*|, \cos\theta) = \min\{B(|v-v_*|, \cos\theta),\, n|v-v_*|^3 \cos^2\theta \,\sin\theta\}. \quad (2.1)$$

Given isotropic initial data $0 \leqslant f^n_0 \in L^1_2(\mathbf{R}^3)$, $f^n_0(v) = f^n_0(|v|)$. By Theorem 3 in ref. 20, the Eq. (BBE) with the kernels B_n has isotropic solutions $f^n(v,t) = f^n(|v|,t)$ in $C^1([0,\infty); L^1(\mathbf{R}^3))$ with $f^n|_{t=0} = f^n_0$:

$$f^n(v,t) = f^n_0(v) + \int_0^t Q_{B_n}(f^n)(v,s)ds, \quad v \in \mathbf{R}^3 \setminus Z_n, \quad t \geqslant 0 \quad (2.2)$$

where Z_n are null sets independent of t. The solutions also conserve the mass and energy:

$$\int_{\mathbf{R}^3} f^n(v,t)(1,|v|^2)dv = \int_{\mathbf{R}^3} f^n_0(v)(1,|v|^2)dv.$$

Using the approximate solutions f^n we now begin our derivation of a weak form of Eq. (BBE). Without loss of generality, the test functions $\varphi(r)$ will be taken as $\varphi(r^2)$ when they appear in integration. We assume that $\varphi \in C^2_b(\mathbf{R}_+)$ so that there are no problem of integrability. Here $C^k_b(\mathbf{R}_+)$ denotes the class of functions $\varphi \in C^k(\mathbf{R}_+)$ satisfying that $\varphi(r)$ and their derivatives $\frac{d^j}{dr^j}\varphi(r)$ up to order k are all bounded on $\mathbf{R}_+$. Let

$$dF^n_t(r) = 4\pi r^2 f^n(r,t)dr.$$

Multiplying $\varphi(|v|^2)$ to both sides of Eq. (2.2) and taking integration we compute with the usual derivation (using Fubini theorem) that

$$\int_{\mathbf{R}_+} \varphi(r^2)dF^n_t(r) = \int_{\mathbf{R}_+} \varphi(r^2)dF^n_0(r) + \int_0^t ds \iint_{\mathbf{R}^2_+} J_{B_n}[\varphi](r,r_*)dF^n_s(r)dF^n_s(r_*)$$

$$+\varepsilon\int_0^t ds\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)\Delta\varphi f^n f^{n\prime} f^{n\prime}_{\ *} d\omega dv dv_*$$

where

$$J_B[\varphi](r,r_*) = \frac{1}{(4\pi)^2}\iiint_{\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2} B(r\sigma - r_*\sigma_*,\omega)[\varphi(|v'|^2)-\varphi(r^2)]d\omega\, d\sigma\, d\sigma_*,$$

$$v' = v - \langle v-v_*,\omega\rangle\omega,\quad v = r\sigma,\quad v_* = r_*\sigma_*,$$
$$\Delta\varphi = \varphi(|v|^2)+\varphi(|v_*|^2)-\varphi(|v'|^2)-\varphi(|v'_*|^2).$$

The main step of our derivation is to compute the collision integral of $f^n f^{n\prime} f^{n\prime}_{\ *}$ with respect to $d\omega dv dv_*$. We show that this integral can be written as an integration with respect to the product measure

$$dF_s^n(r)dF_s^n(r')dF_s^n(r'_*)$$

where r, r', r'_* are independent variables in $\mathbf{R}_+$. For fixed n and s, write $f^n(v,s) = f(|v|)$. Let $x = v - v'$, $y = v - v'_*$. Then (1.3) implies that $x\perp y$ and $|x-y| = |v-v_*| = \sqrt{|x|^2+|y|^2}$. Using Carleman's representation (ref. 3) we have

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)\Delta\varphi f f' f'_* d\omega dv dv_*$$
$$= 2\int_{\mathbf{R}^3} f(|v|)\left[\int_{\mathbf{R}^3}\frac{f(|v-x|)}{|x|^2}\int_{\mathbf{R}^2(x)} B_n(V,\cos\theta)\Delta\varphi\, f(|v-y|)d^\perp y dx\right]dv$$

where $V = \sqrt{|x|^2+|y|^2}$, $\cos\theta = |x|/\sqrt{|x|^2+|y|^2}$, $\theta\in[0,\pi/2]$,

$$\Delta\varphi = \varphi(|v|^2)+\varphi(|v-x|^2+|v-y|^2-|v|^2)-\varphi(|v-x|^2)-\varphi(|v-y|^2),$$

$\mathbf{R}^2(x) = \{y\in\mathbf{R}^3\,|\,y\perp x\}$ for $x\neq 0$, and $d^\perp y$ denotes the 2-dimensional Lebesgue measure on the plan $\mathbf{R}^2(x)$. For $v,x,y\in\mathbf{R}^3$ with $x\neq 0$, $y\neq 0$ and $x\perp y$, consider the decomposition for $B_n = B_n(V,\cos\theta)$:

$$B_n = \frac{|x||v-y|}{|x||v-y|+|y||v-x|}B_n + \frac{|y||v-x|}{|x||v-y|+|y||v-x|}B_n.$$

Then following the proof of Proposition 2 in ref. 20 and noting that $\Delta\varphi$ is symmetric with respect to x and y, we obtain

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v-v_*,\omega)\Delta\varphi f f' f'_* d\omega dv dv_*$$

$$
\begin{aligned}
&= 2\int_{\mathbf{R}^3} f(|v|)\left\{\int_{\mathbf{R}^3} f(|v-x|)\int_{\mathbf{R}^2(x)} \frac{\Delta\varphi}{|x||v-y|+|y||v-x|}\right.\\
&\quad\left.\times\left[\frac{B_n(V,\cos\theta)}{|x|}+\frac{B_n(V,\sin\theta)}{|y|}\right] f(|v-y|)|v-y|d^{\perp}y dx\right\}dv.
\end{aligned}
\tag{2.3}
$$

Let

$$
\begin{aligned}
&\Phi(|v|,|v-x|,|x|,|v-y|,|y|)\\
&=\frac{\Delta\varphi}{|x||v-y|+|y||v-x|}\left[\frac{B_n(V,\cos\theta)}{|x|}+\frac{B_n(V,\sin\theta)}{|y|}\right]
\end{aligned}
$$

and change variable $x=v-z$. Then the inner integral in (2.3) on $d^{\perp}ydx$ is equal to

$$
\int_{\mathbf{R}^3} f(|z|)\int_{\mathbf{R}^2(\xi)} \Phi(|v|,|z|,|v-z|,|v-y|,|y|) f(|v-y|)|v-y|d^{\perp}y\, dz
$$

where $\xi=(v-z)/|v-z|$ and we have used the fact that $\mathbf{R}^2(\xi)=\mathbf{R}^2(\lambda\xi)$ for all $\lambda\in\mathbf{R}\setminus\{0\}$. To compute the inner integral on $d^{\perp}y$ we denote $v_\xi=\langle v,\xi\rangle\xi$. Since $(v-v_\xi)\perp\xi$, i.e., $v-v_\xi\in\mathbf{R}^2(\xi)$, the coordinate transform $y=\rho\omega+v-v_\xi$ in $\mathbf{R}^2(\xi)$ gives

$$
\begin{aligned}
&\int_{\mathbf{R}^2(\xi)} \Phi(|v|,|z|,|v-z|,|v-y|,|y|) f(|v-y|)|v-y|d^{\perp}y\\
&=\int_0^{\infty}\left(\int_{\mathbf{S}^1(\xi)} \Phi\left(|v|,|z|,|v-z|,\sqrt{|v_\xi|^2+\rho^2},|\rho\omega+v-v_\xi|\right)d^{\perp}\omega\right)\\
&\quad\times f\left(\sqrt{|v_\xi|^2+\rho^2}\right)\sqrt{|v_\xi|^2+\rho^2}\;\rho\, d\rho\\
&=\int_{|v_\xi|}^{\infty}\left(\int_{\mathbf{S}^1(\xi)} \Phi\left(|v|,|z|,|v-z|,r'_*,\left|\sqrt{{r'_*}^2-|v_\xi|^2}\,\omega+v-v_\xi\right|\right)d^{\perp}\omega\right)\\
&\quad\times {r'_*}^2 f(r'_*)dr'_*.
\end{aligned}
$$

Here $\mathbf{S}^1(\xi)=\{\omega\in\mathbf{S}^2\,|\,\omega\perp\xi\}$ and $d^{\perp}\omega$ stands for the Lebesgue measure on the circle $\mathbf{S}^1(\xi)$, i.e.,

$$
\int_{\mathbf{S}^1(\xi)} g(\omega)d^{\perp}\omega=\int_0^{2\pi} g(\cos\phi\,\mathbf{i}+\sin\phi\,\mathbf{j})d\phi
$$

where $\mathbf{i}, \mathbf{j}$ are unit vectors such that $\{\mathbf{i}, \mathbf{j}, \xi\}$ forms an orthonormal base of $\mathbf{R}^3$. By calculation using the formula above for Φ on its last variable and noting that $\omega \perp v_\xi$ when $\omega \in \mathbf{S}^1(\xi)$, we obtain (for $r'_* > |v_\xi|$)

$$\int_{\mathbf{S}^1(\xi)} \Phi(\cdots, |\sqrt{{r'_*}^2 - |v_\xi|^2}\,\omega + v - v_\xi|) d^\perp\omega = \int_0^{2\pi} \Phi(\cdots, Y) d\phi$$

where $Y \geqslant 0$,

$$Y^2 = {r'_*}^2 - |v_\xi|^2 + |v|^2 - \langle v, \xi\rangle^2 + 2\sqrt{{r'_*}^2 - |v_\xi|^2}\,\sqrt{|v|^2 - |\langle v, \xi\rangle|^2}\cos\phi.$$

Combining these with spherical coordinate transforms $v = r\sigma$, $z = r'\sigma'$ we obtain

$$\begin{aligned}
&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v - v_*, \omega)\Delta\varphi f f' f'_* d\omega dv dv_* \\
&= 2\iiint_{\mathbf{R}^3_+} \left\{ \iint_{\mathbf{S}^2\times\mathbf{S}^2} d\sigma d\sigma' \int_0^{2\pi} \frac{\Delta\varphi(r, r', r'_*)}{X r'_* + Y r'} \left[\frac{B_n(V, \cos\theta)}{X} + \frac{B_n(V, \sin\theta)}{Y} \right] \right. \\
&\qquad \left. \times \mathbf{1}_{\{r'_* > r|\langle\sigma,\xi\rangle|\}} d\phi \right\} r^2 f(r) {r'}^2 f(r') {r'_*}^2 f(r'_*) dr dr' dr'_* \qquad (2.4)
\end{aligned}$$

where

$$\begin{aligned}
&\Delta\varphi(r, r', r'_*) = \varphi(\hat{r}^2) + \varphi({r'}^2 + {r'_*}^2 - \hat{r}^2) - \varphi({r'}^2) - \varphi({r'_*}^2), \\
&\hat{r} = \min\{r, \sqrt{{r'}^2 + {r'_*}^2}\}, \quad V = \sqrt{X^2 + Y^2}, \\
&\theta = \arccos(X/\sqrt{X^2 + Y^2}); \quad \theta = 0 \quad \text{if} \quad X = Y = 0, \\
&X = |r\sigma - r'\sigma'|, \quad \sigma, \sigma' \in \mathbf{S}^2, \\
&Y = \Big[({r'_*}^2 - r^2\langle\sigma, \xi\rangle^2)^+ + r^2(1 - \langle\sigma, \xi\rangle^2) \\
&\qquad + 2\sqrt{({r'_*}^2 - r^2\langle\sigma, \xi\rangle^2)^+}\; r\sqrt{1 - \langle\sigma, \xi\rangle^2}\cos\phi \Big]^{1/2}, \\
&\xi = \frac{r\sigma - r'\sigma'}{|r\sigma - r'\sigma'|} \quad \text{if} \quad r\sigma \neq r'\sigma'; \qquad \xi = \sigma \quad \text{if} \quad r\sigma = r'\sigma',
\end{aligned}$$

$(a - b)^+ = a - b$ if $a \geqslant b$; $= 0$ if $a < b$, and $\mathbf{1}_{\{\dots\}}$ denotes the indicator: $\mathbf{1}_{\{a > b\}} = 1$ if $a > b$; $= 0$ if $a \leqslant b$. The restriction $\hat{r} = \min\{r, \sqrt{{r'}^2 + {r'_*}^2}\}$ comes from the following relation:

$$r'_* > r|\langle\sigma, \xi\rangle| \quad \Longrightarrow \quad {r'}^2 + {r'_*}^2 > r^2 \qquad (2.5)$$

which is a consequence of the identity

$$r'^2 + r_*'^2 - r^2 = (|r\sigma - r'\sigma'| - r\langle\sigma,\xi\rangle)^2 + r_*'^2 - r^2\langle\sigma,\xi\rangle^2.$$

The following lemma gives a representation of $\Delta\varphi(r, r', r_*')$ which is important for establishing a weak form of Eq. (BBE).

Lemma 1. Let $\varphi \in C^2(\mathbf{R}_+)$ and let $\Delta\varphi(r, r', r_*')$ be defined in above. Then for all $(r, r', r_*') \in \mathbf{R}_+^3$

$$\Delta\varphi(r, r', r_*') = L[D^2\varphi](r, r', r_*')(r'^2 - \hat{r}^2)(r_*'^2 - \hat{r}^2)$$

where $D^2\varphi(r) = \frac{d^2}{dr^2}\varphi(r)$, $\hat{r} = \min\{r\,, \sqrt{r'^2 + r_*'^2}\,\}$,

$$L[D^2\varphi](r, r', r_*') = (1-\lambda)[D^2\varphi](r', r_*') + \lambda\,[D^2\varphi](r, r', r_*'),$$

$$\lambda = \frac{2(r'^2 - \hat{r}^2)(r_*'^2 - \hat{r}^2)}{(r'^2 - \hat{r}^2)^2 + (r_*'^2 - \hat{r}^2)^2} \quad \text{and} \quad \lambda = 0 \quad \text{for} \quad r' = r_*' = \hat{r}, \tag{2.6}$$

$$[D^2\varphi](r', r_*') = \int_0^1 (D^2\varphi)(r_*'^2 + s(r'^2 - r_*'^2))\,ds,$$

$$\begin{aligned}
&[D^2\varphi](r, r', r_*') \\
&= \frac{1}{2}\int_0^1 (1-s)\Big[(D^2\varphi)(r'^2 + s(\hat{r}^2 - r'^2)) + (D^2\varphi)(r_*'^2 + s(\hat{r}^2 - r_*'^2)) \\
&\quad + (D^2\varphi)(r'^2 + s(r_*'^2 - \hat{r}^2)) + (D^2\varphi)(r_*'^2 + s(r'^2 - \hat{r}^2))\Big]ds.
\end{aligned}$$

The function $L[D^2\varphi](r, r', r_*')$ is continuous on $\mathbf{R}_+^3$ and has the bound

$$|L[D^2\varphi](r, r', r_*')| \leqslant 3 \max_{0 \leqslant \rho \leqslant r'^2 + r_*'^2} |D^2\varphi(\rho)|, \quad (r, r', r_*') \in \mathbf{R}_+^3. \tag{2.7}$$

The proof of this lemma will be given in the next section. Let us first introduce a linear functional $K_B[\varphi]$ of φ and a new kernel W_B (for any $B \in C(\mathbf{R}_+ \times [0,1])$):

$$K_B[\varphi](r, r', r_*') = L[D^2\varphi](r, r', r_*')W_B(r, r', r_*'), \qquad \varphi \in C^2(\mathbf{R}_+) \tag{2.8}$$

$$W_B(r,r',r'_*) = \frac{2}{(4\pi)^3}\iint_{\mathbf{S}^2\times\mathbf{S}^2} d\sigma d\sigma' \int_0^{2\pi} W_B(r,r',r'_*,\sigma,\sigma',\phi)d\phi \quad (2.9)$$

where (with notations collected under Eq. (2.4))

$$W_B(r,r',r'_*,\sigma,\sigma',\phi) = \frac{(r'^2-r^2)(r'^2_*-r^2)}{Xr'_*+Yr'}\left[\frac{B(V,\cos\theta)}{X}+\frac{B(V,\sin\theta)}{Y}\right]\mathbf{1}_{\{r'_*>r|\langle\sigma,\xi\rangle|\}}$$

for $(r'-r)(r'_*-r)\neq 0$, and

$$W_B(r,r',r'_*,\sigma,\sigma',\phi)=0 \quad \text{for} \quad (r'-r)(r'_*-r)=0.$$

It is easily proven that if $(r'-r)(r'_*-r)\neq 0$ and $r'_*>r|\langle\sigma,\xi\rangle|$, then

$$X \geqslant |r'-r|>0, \qquad Y \geqslant |r'_*-r|>0$$

so the function $W_B(r,r',r'_*,\sigma,\sigma',\phi)$ is well defined. Also from the relation (2.5) we have

$$(r'^2-r^2)(r'^2_*-r^2)\mathbf{1}_{\{r'_*>r|\langle\sigma,\xi\rangle|\}} = (r'^2-\hat{r}^2)(r'^2_*-\hat{r}^2)\mathbf{1}_{\{r'_*>r|\langle\sigma,\xi\rangle|\}}.$$

This gives (at least for $(r'-r)(r'_*-r)\neq 0$) a detailed representation of $K_B[\varphi]$:

$$K_B[\varphi](r,r',r'_*) = \frac{2}{(4\pi)^3}\iint_{\mathbf{S}^2\times\mathbf{S}^2} d\sigma d\sigma' \int_0^{2\pi} \frac{\Delta\varphi(r,r',r'_*)}{Xr'_*+Yr'}\left[\frac{B(V,\cos\theta)}{X}+\frac{B(V,\sin\theta)}{Y}\right] \times\mathbf{1}_{\{r'_*>r|\langle\sigma,\xi\rangle|\}}d\phi. \quad (2.10)$$

In particular for the hard sphere model $B(V,\tau)=V\tau$ we have

$$K_B[\varphi](r,r',r'_*) = \frac{4}{(4\pi)^3}\iint_{\mathbf{S}^2\times\mathbf{S}^2} d\sigma d\sigma' \int_0^{2\pi} \frac{\Delta\varphi(r,r',r'_*)}{Xr'_*+Yr'}\mathbf{1}_{\{r'_*>r|\langle\sigma,\xi\rangle|\}}d\phi.$$

It is also useful to write $J_B[\varphi](r,r_*)$ in a detailed version:

$$J_B[\varphi](r,r_*) = \frac{2}{(4\pi)^2}\iint_{\mathbf{S}^2\times\mathbf{S}^2} d\sigma d\sigma_* \int_0^{\pi/2} B(|r\sigma-r_*\sigma_*|,\cos\theta)\sin\theta$$

$$\times \int_0^{2\pi} [\varphi(|v'|^2) - \varphi(r^2)] d\phi \, d\theta \tag{2.11}$$

where

$$|v'|^2 = r^2 \sin^2\theta + r_*^2 \cos^2\theta - 2rr_* \sin\theta\cos\theta\sqrt{1-\langle\sigma,\sigma_*\rangle^2}\cos\phi.$$

Now if we replace the density $4\pi r^2 f(r)dr$ with a positive Borel measure $dF(r)$ on $\mathbf{R}_+$ satisfying $\int_{\mathbf{R}_+}(1+r^2)dF(r)<\infty$, then from Eqs. (2.4), (2.10), and (2.11) we see that the collision integral operator (1.2) for the BBE model in a weak form can be defined as a linear functional $\varphi \mapsto \langle Q_B(F), \varphi\rangle$ on $\varphi \in C_b^2(\mathbf{R}_+)$ by

$$\langle Q_B(F), \varphi\rangle := \iint_{\mathbf{R}_+^2} J_B[\varphi](r, r_*) dF(r) dF(r_*) + \varepsilon \iiint_{\mathbf{R}_+^3} K_B[\varphi](r, r', r_*') dF(r) dF(r') dF(r_*').$$

Later we will prove that (see Lemma 1, Lemma 2, Eqs. (3.13) and (3.15) in Section 3) the functions $J_B[\varphi](r, r_*)$ and $K_B[\varphi](r, r', r_*')$ are continuous on $(r, r_*) \in \mathbf{R}_+^2$ and on $(r, r', r_*') \in \mathbf{R}_+^3$ respectively for all $\varphi \in C_b^2(\mathbf{R}_+)$, and satisfy

$$\iint_{\mathbf{R}_+^2} |J_B[\varphi](r, r_*)| dF(r) dF(r_*) \leqslant C\|\varphi\|_{L^\infty}(\|F\|_2)^2,$$
$$\iiint_{\mathbf{R}_+^3} |K_B[\varphi](r, r', r_*')| dF(r) dF(r') dF(r_*') \leqslant C\|D^2\varphi\|_{L^\infty}(\|F\|_2)^3$$

where $\|F\|_2 = \int_{\mathbf{R}_+}(1+r^2)dF(r)$, $C = C_0 \sup_{V\geqslant 0, \tau\in[0,1]} \frac{B(V,\tau)}{1+V}$, and C_0 is an absolute constant. Therefore the linear functional $\varphi \mapsto \langle Q_B(F), \varphi\rangle$ is well defined and bounded on $C_b^2(\mathbf{R}_+)$.

From the above derivation we now obtain the following weak form for the approximate solutions f^n of Eq. (BBE) (with the kernel B_n): For all $\varphi \in C_b^2(\mathbf{R}_+)$,

$$\int_{\mathbf{R}_+} \varphi(r^2) dF_t^n(r) = \int_{\mathbf{R}_+} \varphi(r^2) dF_0^n(r) + \int_0^t \langle Q_{B_n}(F_s^n), \varphi\rangle \, ds, \quad t \geqslant 0$$

$(n = 1, 2, 3, \ldots)$. Therefore taking weak limit leads to the following

Definition of Isotropic Distributional Solutions: Let B a collision kernel satisfying the conditions $(i)-(ii)$. Let F_0 be a positive Borel measure on $\mathbf{R}_+$ with $\int_{\mathbf{R}_+}(1+r^2)dF_0(r)<\infty$. If a family $\{F_t\}_{t\geqslant 0}$ of positive Borel measures on $\mathbf{R}_+$ satisfies

(1) $F_t|_{t=0}=F_0$,

(2) $\sup_{t\in[0,T]}\int_{\mathbf{R}_+}(1+r^2)dF_t(r)<\infty \qquad \forall\, 0<T<\infty$,

(3) the function $t\mapsto \int_{\mathbf{R}_+}\varphi(r^2)dF_t(r)$ is in $C^1([0,\infty))$ for all $\varphi\in C^2_b(\mathbf{R}_+)$, and

(4) the equation

$$\frac{d}{dt}\int_{\mathbf{R}_+}\varphi(r^2)dF_t(r)=\langle Q_B(F_t),\varphi\rangle \tag{2.12}$$

holds for all $\varphi\in C^2_b(\mathbf{R}_+)$ and all $t\geqslant 0$, then F_t is called an isotropic distributional solution to Eq. (BBE) with the initial datum F_0. Furthermore, if F_t conserves the mass and energy, i.e., if for all $t\geqslant 0$

$$\int_{\mathbf{R}_+}dF_t(r)=\int_{\mathbf{R}_+}dF_0(r),\qquad \int_{\mathbf{R}_+}r^2dF_t(r)=\int_{\mathbf{R}_+}r^2dF_0(r),$$

then we say that F_t is a conservative solution. ∎

Note that since the constant function $\varphi\equiv 1$ gives $J_B[1]=K_B[1]=0$, the conservation of mass holds for all solutions. But the conservation of energy does not hold automatically. In general, we can only prove that the energy $t\mapsto\int_{\mathbf{R}_+}r^2dF_t(r)$ is non-decreasing (see Section 4). We note also that in our Definition, the test functions φ are independent of t. But this does not lose generality: It is easily shown that if F_t is an isotropic distributional solution, then

$$\frac{d}{dt}\int_{\mathbf{R}_+}\varphi(r^2,t)dF_t(r)=\int_{\mathbf{R}_+}\frac{\partial}{\partial t}\varphi(r^2,t)dF_t(r)+\langle Q_B(F_t),\varphi(\cdot,t)\rangle \tag{2.13}$$

for all $\varphi\in C^2_b(\mathbf{R}^2_+)$ and all $t\geqslant 0$. Here $C^k_b(\mathbf{R}^2_+)$ denotes the class of functions $\varphi\in C^k(\mathbf{R}^2_+)$ satisfying that $\varphi(r,t)$ and $\frac{\partial^{i+j}}{\partial r^i\,\partial t^j}\varphi(r,t)$ with order $i+j\leqslant k$ are all bounded on $\mathbf{R}^2_+$. In fact, for any $\varphi\in C^2_b(\mathbf{R}^2_+)$ and any $t_0\in[0,\infty)$, applying Definition to the test function $r\mapsto\varphi(r,t_0)$ we have for all $t\geqslant 0$ with $t\neq t_0$

$$\tfrac{1}{t-t_0}\left(\int_{\mathbf{R}_+}\varphi(r^2,t)dF_t(r)-\int_{\mathbf{R}_+}\varphi(r^2,t_0)dF_{t_0}(r)\right)$$

$$= \int_{\mathbf{R}_+} \tfrac{1}{t-t_0} \left(\int_{t_0}^{t} \tfrac{\partial}{\partial s} \varphi(r^2, s) ds \right) dF_t(r) + \tfrac{1}{t-t_0} \int_{t_0}^{t} \langle Q_B(F_s), \varphi(\cdot, t_0) \rangle \, ds$$
$$\to \int_{\mathbf{R}_+} \tfrac{\partial}{\partial t} \varphi(r^2, t_0) dF_{t_0}(r) + \langle Q_B(F_{t_0}), \varphi(\cdot, t_0) \rangle \quad \text{as} \quad t \to t_0.$$

This gives the general weak form (2.13). A remark on the validity of this derivation is given at the end of Section 4.

In the above formulations there are also many other crucial details such as integrability and continuity (in pointwise and weak topology) that should be proven in the rigorous sense. These are the main content of the next section.

Now we shall prove the necessity of the condition $B(V, 0) = 0$ for kernels B as mentioned in the Introduction.

Proposition 2. Let $B(V, \tau)$ be continuous on $\mathbf{R}_+ \times [0, 1]$ such that $K_B[\varphi](r, r', r'_*)$ is continuous on $\mathbf{R}^3_+$ for all $\varphi \in C_b^2(\mathbf{R}_+)$. Then $B(V, 0) \equiv 0$.

Proof. Given any $V > 0$. Choose for instance $r_n = 0, r'_n = V, r'_{*n} = \epsilon_n > 0$ ($n = 1, 2, ...$) and $\epsilon_n \to 0$ ($n \to \infty$). Then by continuity of $K_B[\varphi]$ and the definition of W_B we have

$$\lim_{n \to \infty} K_B[\varphi](0, V, \epsilon_n) = K_B[\varphi](0, V, 0) = 0.$$

On the other hand we compute $X_n = V, Y_n = \epsilon_n, V_n = \sqrt{V^2 + \epsilon_n^2}$, and $\sin\theta_n = Y_n / V_n \to 0$ so by the continuity of $L[D^2\varphi]$ and B we have

$$\begin{aligned} K_B[\varphi](0, V, \epsilon_n) &= L[D^2\varphi](0, V, \epsilon_n) W_B(0, V, \epsilon_n) \\ &= \frac{1}{2} L[D^2\varphi](0, V, \epsilon_n) \left[\epsilon_n B(V_n, \cos\theta_n) + V B(V_n, \sin\theta_n)\right] \\ &\to \frac{1}{2} L[D^2\varphi](0, V, 0) V B(V, 0) \quad (n \to \infty). \end{aligned}$$

Thus

$$L[D^2\varphi](0, V, 0) B(V, 0) = 0 \qquad \forall V > 0, \quad \forall \varphi \in C_b^2(\mathbf{R}_+).$$

Since

$$L[D^2\varphi](0, V, 0) = \int_0^1 (D^2\varphi)(sV^2) ds = \frac{1}{V^2} \left(\frac{d\varphi}{dr}(V^2) - \frac{d\varphi}{dr}(0) \right)$$

which can be non-zero for some $\varphi \in C_b^2(\mathbf{R}_+)$ (e.g., $\varphi(r) = e^{-r}$), it follows that $B(V, 0) \equiv 0$. ∎

Finally let us prove the divergence (1.10) in Proposition 1. First of all we note that the boundedness of the function Ψ in (1.10) implies the strong divergence

$$\lim_{n\to\infty}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)f_n(v)f_n(v')f_n(v'_*)d\omega dv_*dv=\infty. \quad (2.14)$$

An observation that leads to the divergence (2.14) is the following formulation using Dirac δ-functions: Let $\delta_0(v)=\delta_0(|v|)\geqslant 0$ be a Dirac δ-function concentrated at $v=0$ with $\int_{\mathbf{R}^3}\delta_0(v)dv=1$. Let $0<V_0<\infty, 0<\tau_0<1$ be given in Proposition 1. Choose $x_0, y_0\in\mathbf{R}^3\setminus\{0\}$ satisfying $x_0\perp y_0$ and $|x_0|=V_0\tau_0$, $|y_0|=V_0\sqrt{1-\tau_0^2}$ so that $|x_0-y_0|=V_0$, $|x_0|/|x_0-y_0|=\tau_0$. Consider $f(v)=\delta_0(v)+\delta_0(v-x_0)+\delta_0(v-y_0)$. By nonnegativity and Carleman representation (this time we use $(v',v'_*)=(v+x,v+y)$) we have formally that

$$\begin{aligned}
Q &:= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)f(v)f(v')f(v'_*)d\omega dv_*dv\\
&\geqslant \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\delta_0(v)\delta_0(v'-x_0)\delta_0(v'_*-y_0)d\omega dv_*dv\\
&= 2\int_{\mathbf{R}^3}\delta_0(v)\int_{\mathbf{R}^3}\frac{\delta_0(v+x-x_0)}{|x|^2}\\
&\qquad\times\left(\int_{\mathbf{R}(x)}\delta_0(v+y-y_0)B(|x-y|,|x|/|x-y|)d^{\perp}y\right)dx\\
&= 2\int_{\mathbf{R}^3}\frac{\delta_0(x-x_0)}{|x|^2}\left(\int_{\mathbf{R}(x)}\delta_0(y-y_0)B(|x-y|,|x|/|x-y|)d^{\perp}y\right)dx\\
&= \frac{2}{|x_0|^2}B(|x_0-y_0|,|x_0|/|x_0-y_0|)\int_{\mathbf{R}^2(x_0)}\delta_0(y-y_0)d^{\perp}y\,.
\end{aligned}$$

Since $y_0\in\mathbf{R}^2(x_0)$, the translation invariance on $\mathbf{R}^2(x_0)$ gives

$$\int_{\mathbf{R}^2(x_0)}\delta_0(y-y_0)d^{\perp}y=\int_{\mathbf{R}^2}\delta_0(y)dy=\frac{1}{2}\int_{\mathbf{R}^3}\frac{\delta_0(y)}{|y|}dy=\infty.$$

Since $B(|x_0-y_0|,|x_0|/|x_0-y_0|)=B(V_0,\tau_0)>0$, it follows that $Q=\infty$. ∎

Proof of Proposition 1. The proof of (1.10) follows the same idea as above but details are technically different because it should be careful to deal with the multiplication of (approximate) δ-functions. Let $\Phi(r)=$

$\exp(-\frac{1}{1-r^2})$ if $0 \leqslant r < 1$; $\Phi(r) = 0$ if $r \geqslant 1$. Let V_0, τ_0, x_0 and y_0 be given in the above argument. For any $n \geqslant 1$, consider

$$f_n^{(1)}(v) = n^9 \Phi(n^3|v|),\ \ f_n^{(2)}(v) = n^6 \Phi(n^2|v - x_0|),\ \ f_n^{(3)}(v) = n^3 \Phi(n|v - y_0|),$$

$$f_n(v) = f_n^{(1)}(v) + f_n^{(2)}(v) + f_n^{(3)}(v).$$

Then $0 \leqslant f_n \in C_c^\infty(\mathbf{R}^3)$ and

$$\int_{\mathbf{R}^3} f_n(v)dx = 3\int_{\mathbf{R}^3} \Phi(|v|)dv, \quad n = 1, 2, \dots .$$

Let $\Psi(v, v', v'_*)$ be a function satisfying the condition in Proposition 1. Since $f_n(v) f_n(v') f_n(v'_*) \geqslant f_n^{(1)}(v) f_n^{(2)}(v') f_n^{(3)}(v'_*)$, using Carleman representation we have

$$\begin{aligned}
Q_n &:= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)|\Psi(v, v', v'_*)| f_n(v) f_n(v') f_n(v'_*) d\omega dv_* dv \\
&\geqslant \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v - v_*, \omega)|\Psi(v, v', v'_*)| f_n^{(1)}(v) f_n^{(2)}(v') f_n^{(3)}(v'_*) d\omega dv_* dv \\
&= 2\int_{\mathbf{R}^3} f_n^{(1)}(v) \int_{\mathbf{R}^3} \frac{f_n^{(2)}(v+x)}{|x|^2} \int_{\mathbf{R}^2(x)} f_n^{(3)}(v+y) \\
&\qquad \times B(|x-y|, |x|/|x-y|)|\Psi(v, v+x, v+y)| d^\perp y\, dx dv \\
&= 2n^{18} \int_{\mathbf{R}^3} \Phi(n^3|v|) \int_{\mathbf{R}^3} \frac{\Phi(n^2|v+x-x_0|)}{|x|^2} \int_{\mathbf{R}^2(x)} \Phi(n|v+y-y_0|) \\
&\qquad \times B(|x-y|, |x|/|x-y|)|\Psi(v, v+x, v+y)| d^\perp y\, dx dv \\
&= 2n^9 \int_{|v| \leqslant 1} \Phi(|v|) \int_{\mathbf{R}^3} \frac{\Phi(n^2|v/n^3 + x - x_0|)}{|x|^2} \int_{\mathbf{R}^2(x)} \Phi(n|v/n^3 + y - y_0|) \\
&\qquad \times B(|x-y|, |x|/|x-y|)|\Psi(v/n^3, v/n^3 + x, v/n^3 + y)| d^\perp y\, dx dv.
\end{aligned}$$

By the assumption on Ψ, $|\Psi(0, x_0, y_0)| > 0$. So there exists $\eta_0 > 0$ such that

$$A_0^* := \min\{|\Psi(v, v', v'_*)| \mid |v| \leqslant \eta_0,\ |v' - x_0| \leqslant \eta_0,\ |v'_* - y_0| \leqslant \eta_0\} > 0.$$

Let $0 < \eta < \min\{V_0,\ \tau_0,\ 1 - \tau_0\}$ be small enough such that

$$B_0^* := \inf\{B(V, \tau) \mid |V - V_0| \leqslant \eta,\ |\tau - \tau_0| \leqslant \eta\} > 0.$$

Let $n > n_0 := \max\{2/|x_0|, 6/V_0, 3/\eta, 8/(\eta V_0), 1/\eta_0\}$. Since $\Phi(r)$ is supported on $[0,1]$, it follows that for all (v,x,y) satisfying $|v| \leqslant 1$ and $\Phi(n^2|v/n^3+x-x_0|)\Phi(n|v/n^3+y-y_0|)\neq 0$ we have $|v/n^3+x-x_0| \leqslant 1/n^2$, $|v/n^3+y-y_0| \leqslant 1/n$, $|x-x_0| \leqslant 1/n$, $|y-y_0| \leqslant 2/n$, $|x| \leqslant 2|x_0|$, $||x-y|-V_0| = ||x-y|-|x_0-y_0|| \leqslant 3/n \leqslant \eta$ and $||x|/|x-y|-\tau_0| = ||x|/|x-y|-|x_0|/|x_0-y_0|| \leqslant 8/(V_0 n) \leqslant \eta$, and so

$$|\Psi(v/n^3, v/n^3+x, v/n^3+y)| \geqslant A_0^*, \quad B(|x-y|, |x|/|x-y|) \geqslant B_0^*.$$

Also, since $\Phi(r)$ is non-increasing, it follows that for all $|v| \leqslant 1$,

$$\begin{aligned}\Phi(n^2|v/n^3+x-x_0|) &\geqslant \Phi(1/n+n^2|x-x_0|),\\ \Phi(n|v/n^3+y-y_0|) &\geqslant \Phi(1/n^2+n|y-y_0|).\end{aligned}$$

Thus for the constant $C_0^* = \|\Phi\|_{L^1} A_0^* B_0^*/(2|x_0|)^2 > 0$, we obtain

$$\begin{aligned}Q_n &\geqslant C_0^* n^9 \int_{\mathbf{R}^3} \Phi(1/n+n^2|x-x_0|) \int_{\mathbf{R}^2(x)} \Phi(1/n^2+n|y-y_0|) d^\perp y\, dx\\ &= C_0^* n^3 \int_{|x| \leqslant 1} \Phi(1/n+|x|) \int_{\mathbf{R}^2(x/n^2+x_0)} \Phi(1/n^2+n|y-y_0|) d^\perp y dx.\end{aligned}$$

Let

$$z_n(x) = y_0 - \frac{\langle y_0, x/n^2\rangle}{|x_0|^2+\langle x_0, x/n^2\rangle} x_0, \qquad |x| \leqslant 1.$$

Then the condition $\langle x_0, y_0\rangle = 0$ implies that

$$\langle z_n(x), x/n^2+x_0\rangle = 0, \quad i.e., \quad z_n(x) \in \mathbf{R}^2(x/n^2+x_0).$$

Furthermore for all $n > n_0$ and all $|x| \leqslant 1$ we have $|y-y_0| \leqslant |y-z_n(x)| + |z_n(x)-y_0| \leqslant |y-z_n(x)| + c/n^2$ and so $\Phi(1/n^2+n|y-y_0|) \geqslant \Phi(1/n^2 + c/n + n|y-z_n(x)|)$ where c depends only x_0 and y_0. Let $\epsilon_n = \max\{1/n, 1/n^2+c/n\}$. Then using translation invariance on $\mathbf{R}^2(x/n^2+x_0)$ and noting that $C_0^* > 0$ and $\epsilon_n \to 0$ we obtain

$$\begin{aligned}Q_n &\geqslant C_0^* n^3 \int_{|x| \leqslant 1} \Phi(\epsilon_n+|x|) \int_{\mathbf{R}^2(x/n^2+x_0)} \Phi(\epsilon_n+n|y-z_n(x)|) d^\perp y dx\\ &= C_0^* n^3 \int_{\mathbf{R}^3} \Phi(\epsilon_n+|x|) dx \cdot \int_{\mathbf{R}^2} \Phi(\epsilon_n+n|y|) dy\\ &= C_0^* \cdot n \cdot \int_{\mathbf{R}^3} \Phi(\epsilon_n+|x|) dx \cdot \int_{\mathbf{R}^2} \Phi(\epsilon_n+|y|) dy \to \infty \quad (n\to\infty). \quad \blacksquare\end{aligned}$$

3. SOME LEMMAS

Throughout this section, unless otherwise stated, we always assume that the kernel B satisfies the conditions (i)-(ii). We first prove Lemma 1 stated in the above section.

Proof of Lemma 1. First of all, the estimate (2.7) is obvious by definition of $L[D^2\varphi]$. Now we prove the representation (2.8). Let $\varphi \in C^2(\mathbf{R}_+)$ and let $r, r', r'_* \geqslant 0$. We may assume that $(r'^2 - \hat{r}^2)^2 + (r'^2_* - \hat{r}^2)^2 > 0$ (otherwise (2.8) holds obviously). By definition of $L[D^2\varphi]$ we have $L[D^2\varphi](r, r', r'_*) \equiv L[D^2\varphi](\hat{r}, r', r'_*)$. Since $r'^2 + r'^2_* \geqslant \hat{r}^2$ we may assume for notational convenience that $r'^2 + r'^2_* \geqslant r^2$. By Taylor formula we have

$$\varphi(r^2) - \varphi(r'^2) = \varphi_1(r'^2)\,(r^2 - r'^2) + \varphi_2(r, r')\,(r^2 - r'^2)^2, \tag{3.1}$$

$$\varphi(r^2) - \varphi(r'^2_*) = \varphi_1(r'^2_*)\,(r^2 - r'^2_*) + \varphi_2(r, r'_*)\,(r^2 - r'^2_*)^2, \tag{3.2}$$

$$\varphi(r'^2 + r'^2_* - r^2) - \varphi(r'^2) = \varphi_1(r'^2)\,(r'^2_* - r^2) + \varphi_3(r, r', r'_*)\,(r'^2_* - r^2)^2, \tag{3.3}$$

$$\varphi(r'^2 + r'^2_* - r^2) - \varphi(r'^2_*) = \varphi_1(r'^2_*)\,(r'^2 - r^2) + \varphi_3(r, r'_*, r')\,(r'^2 - r^2)^2 \tag{3.4}$$

where $\varphi_1(r) = D\varphi(r) = \frac{d}{dr}\varphi(r)$,

$$\varphi_2(r, r') = \int_0^1 (1-s)(D^2\varphi)(r'^2 + s(r^2 - r'^2))ds,$$

$$\varphi_3(r, r', r'_*) = \int_0^1 (1-s)(D^2\varphi)(r'^2 + s(r'^2_* - r^2))ds.$$

Let

$$\alpha_1 := \frac{(r^2 - r'^2_*)^2}{(r^2 - r'^2)^2 + (r^2 - r'^2_*)^2}, \qquad \alpha_2 := \frac{(r^2 - r'^2)^2}{(r^2 - r'^2)^2 + (r^2 - r'^2_*)^2}.$$

Multiplying α_1 to equations (3.1), (3.4) and α_2 to (3.2), (3.3) respectively, and denoting

$$\varphi_4(r', r'_*) = \int_0^1 (D^2\varphi)(r'^2_* + s(r'^2 - r'^2_*))ds$$

we compute

$$
\begin{aligned}
&\Delta\varphi(r, r', r*')\\
&=\alpha_1[\varphi_1(r'^2)-\varphi_1(r_*'^2)](r^2-r'^2)+\alpha_2[\varphi_1(r_*'^2)-\varphi_1(r'^2)](r^2-r_*'^2)\\
&\quad+\alpha_1\,\varphi_2(r,r')\,(r^2-r'^2)^2+\alpha_2\,\varphi_2(r,r_*')\,(r^2-r_*'^2)^2\\
&\quad+\alpha_2\,\varphi_3(r,r',r_*')\,(r_*'^2-r^2)^2+\alpha_1\,\varphi_3(r,r_*',r')\,(r'^2-r^2)^2\\
&=\alpha_1\,\varphi_4(r',r_*')\,(r'^2-r_*'^2)(r^2-r'^2)-\alpha_2\,\varphi_4(r',r_*')\,(r'^2-r_*'^2)(r^2-r_*'^2)\\
&\quad+\alpha_1\,\varphi_2(r,r')\,(r^2-r'^2)^2+\alpha_2\,\varphi_2(r,r_*')\,(r^2-r_*'^2)^2\\
&\quad+\alpha_2\,\varphi_3(r,r',r_*')\,(r_*'^2-r^2)^2+\alpha_1\,\varphi_3(r,r_*',r')\,(r'^2-r^2)^2\\
&=(r'^2-r^2)(r_*'^2-r^2)\Big\{(1-\lambda)\varphi_4(r',r_*')\\
&\quad+\lambda\cdot\frac{1}{2}\Big[\varphi_2(r,r')+\varphi_2(r,r_*')+\varphi_3(r,r',r_*')+\varphi_3(r,r_*',r')\Big]\Big\}\\
&=(r'^2-r^2)(r_*'^2-r^2)L[D^2\varphi](r,r',r_*')
\end{aligned}
$$

where λ is given by (2.6). This proves (2.8).

Next we prove the continuity of $L[D^2\varphi](r,r',r_*')$ on $\mathbf{R}_+^3$. Because the restriction $\hat{r}=\min\{r, \sqrt{r'^2+r_*'^2}\}$ is continuous, we need only to prove that $L[D^2\varphi](r,r',r_*')$, being as a function defined on the set $S:=\{(r,r',r_*')\in\mathbf{R}_+^3\,|\,r'^2+r_*'^2\geqslant r^2\}$, is continuous on S. It is obvious that this function is continuous at points $(r,r',r_*')\in S$ satisfying $(r'^2-r^2)^2+(r_*'^2-r^2)^2>0$. So we need only to prove the continuity at $r=r'=r_*'$. Let $(\rho,\rho',\rho_*')\in S$. If $(\rho^2-\rho'^2)^2+(\rho^2-\rho_*'^2)^2>0$, then by definition of λ in (2.6) we have $|\lambda|\leqslant 1$ which gives

$$
\begin{aligned}
&\Big|L[D^2\varphi](\rho,\rho',\rho_*')-L[D^2\varphi](r,r,r)\Big|=\Big|L[D^2\varphi](\rho,\rho',\rho_*')-\varphi_4(r,r)\Big|\\
&\leqslant\ \big|\varphi_4(\rho',\rho_*')-\varphi_4(r,r)\big|\\
&\quad+\frac{1}{2}\Big|\varphi_2(\rho,\rho')+\varphi_2(\rho,\rho_*')+\varphi_3(\rho,\rho',\rho_*')+\varphi_3(\rho,\rho_*',\rho')-2\varphi_4(\rho',\rho_*')\Big|.
\end{aligned}
$$

This inequality holds also for $\rho=\rho'=\rho_*'$. Since the right hand-side of this inequality tends to zero as $(\rho,\rho',\rho_*')\to(r,r,r)$, it follows that the function $L[D^2\varphi](r,r',r_*')$ is continuous at (r,r,r). Thus $L[D^2\varphi](r,r',r_*')$ is continuous on S and therefore on $\mathbf{R}_+^3$.

Lemma 2. Let $W_B(r, r', r'_*)$ be given in (2.9). Then W_B is continuous on $\mathbf{R}^3_+$ and satisfies

$$|W_B(r, r', r'_*)| \leqslant C(r' + r'_*)(1 + r' + r'_*), \qquad (r, r', r_*) \in \mathbf{R}^3_+, \tag{3.5}$$

$$\frac{|W_B(r, r', r'_*)|}{(1+r^2)(1+r'^2)(1+r'^2_*)} \to 0 \qquad \text{as} \quad r^2 + r'^2 + r'^2_* \to \infty \tag{3.6}$$

where $C = C_0 \sup_{V \geqslant 0,\, \tau \in [0,1]} \frac{B(V,\tau)}{1+V}$, C_0 is an absolute constant.

Remark 1. The solo estimate (3.5) holds in fact for any $B(V, \tau)$ that satisfies the first condition in (ii); the collision operator $Q_B(F)$ as a linear functional can be still defined and bounded on $C^2_b(\mathbf{R}_+)$. While the continuity of W_B is mainly used to applying the Dini's monotone uniform convergence on compact domains for continuous functions (see the proof of Lemma 3). This uniform convergence together with the decay (3.6) enable us to prove the weak convergence (in measure spaces) of approximate solutions of Eq. (BBE).

Proof of Lemma 2. We first prove the estimate (3.5). Recall definition of W_B. It suffices to prove that

$$|W_B(r, r', r'_*, \sigma, \sigma', \phi)| \leqslant 27(r' + r'_*)\,[B(V, \cos\theta)\sin\theta + B(V, \sin\theta)\cos\theta] \tag{3.7}$$

for all $(r, r', r'_*) \in \mathbf{R}^3_+$ and all $(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$. We can assume that $W_B(r, r', r'_*, \sigma, \sigma', \phi) \neq 0$. This implies that $r'_* > r|\langle\sigma', \xi\rangle|$, $X \geqslant |r' - r| > 0$, $Y \geqslant |r'_* - r| > 0$ and $r'^2 + r'^2_* > r^2$. Since $X = V\cos\theta$, $Y = V\sin\theta$, these imply

$$|W_B(r, r', r'_*, \sigma, \sigma', \phi)| \leqslant \frac{(r'+r)(r'_*+r)}{Xr'_* + Yr'}\,[Y B(V, \cos\theta) + X B(V, \sin\theta)]$$
$$= \frac{(r'+r)(r'_*+r)}{(\cos\theta)r'_* + (\sin\theta)r'}\,[B(V, \cos\theta)\sin\theta + B(V, \sin\theta)\cos\theta].$$

(1) If $r' > r/2$ and $r'_* > r/2$, then

$$\frac{(r'+r)(r'_*+r)}{(\cos\theta)r'_* + (\sin\theta)r'} \leqslant \frac{9r'r'_*}{\min\{r', r'_*\}} \leqslant 9(r' + r'_*).$$

(2) If $r' \leqslant r/2$ and $r'_* > r/2$, then $X \geqslant |r' - r| = r - r' \geqslant r/2$ and so $r' + r \leqslant \frac{3}{2}r \leqslant 3X$. Also $r'_* + r \leqslant 3r'_*$. This implies that

$$\frac{(r'+r)(r'_*+r)}{(\cos\theta)r'_* + (\sin\theta)r'} \leqslant \frac{3X \cdot 3r'_*}{(\cos\theta)r'_*} = 9V \leqslant 27(r'+r'_*).$$

Here we have used obvious inequalities $V \leqslant X + Y$ and $X \leqslant r + r', Y \leqslant r + r'_*$.

(3) Similarly if $r' > r/2$ and $r'_* \leqslant r/2$, then

$$\frac{(r'+r)(r'_*+r)}{(\cos\theta)r'_* + (\sin\theta)r'} \leqslant \frac{3Y \cdot 3r'}{(\sin\theta)r'} = 9V \leqslant 27(r'+r'_*).$$

This proves (3.7) and thus (3.5) holds. Next we prove the limit (3.6). Let

$$A(\delta) = \sup_{V \geqslant 0,\, 0 \leqslant \tau \leqslant \delta} \frac{B(V,\tau)}{1+V}, \qquad 0 < \delta < 1.$$

By assumption on B we have $A(\delta) \to 0$ as $\delta \to 0+$. We now prove the following estimate which implies (3.6): For any $R > 4$

$$|W_B(r, r', r'_*, \sigma, \sigma', \phi)| \leqslant C\left(A\left(\frac{1}{\sqrt{R}}\right) + \frac{1}{\sqrt{R}}\right)(1+r^2)(1+r'^2)(1+r'^2_*) \tag{3.8}$$

for all $(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$ and all $r^2 + r'^2 + r'^2_* > 16R^2$. Here and below the constants C are as given in the lemma. As above, we need only to consider the case $W_B(r, r', r'_*, \sigma, \sigma', \phi) \neq 0$. Suppose $R > 4$ and $r^2 + r'^2 + r'^2_* > 16R^2$. In what follows we assume without loss of generality that $r'_* \geqslant r'$.

(1) Suppose that $r > \sqrt{R}$ or $r' \geqslant 2\sqrt{R}$. Then by the estimate (3.7)

$$|W_B(r, r', r'_*, \sigma, \sigma', \phi)| \leqslant C(r'+r'_*)(1+r'+r'_*)$$

$$\leqslant \frac{C(1+r^2)(1+r'^2)(1+r'^2_*)}{(1+r^2)(1+r'^2)} \leqslant \frac{C}{R}(1+r^2)(1+r'^2)(1+r'^2_*).$$

(2) Suppose that $r \leqslant \sqrt{R}$ and $r' < 2\sqrt{R}$. Then $r'_* \geqslant \sqrt{16R^2 - 5R} > 3R + \sqrt{R}$. This implies that $Y \geqslant r'_* - r > 3R$ and $X \leqslant r + r' \leqslant 3\sqrt{R}$. Thus $\cos\theta \leqslant X/Y \leqslant 1/\sqrt{R}$ and therefore by (3.7) we get

$$\begin{aligned}|W_B(r, r', r'_*, \sigma, \sigma', \phi)| &\leqslant 54 r_*(1+V)\left(A\left(\frac{1}{\sqrt{R}}\right) + C\frac{1}{\sqrt{R}}\right)\\ &\leqslant C(1+{r'_*}^2)\left(A\left(\frac{1}{\sqrt{R}}\right) + \frac{1}{\sqrt{R}}\right).\end{aligned}$$

This proves the estimate (3.8).

Now we are going to prove the continuity of $W_B(r, r', r'_*)$. Given any $(r, r', r'_*) \in \mathbf{R}^3_+$. Let $\{(r_n, r'_n, r'_{*n})\}_{n=1}^{\infty}$ be any sequence in $\mathbf{R}^3_+$ satisfying $(r_n, r'_n, r'_{*n}) \to (r, r', r'_*)(n \to \infty)$. From the estimate (3.7) one sees that the sequence $\{W_B(r_n, r'_n, r'_{*n}, \sigma, \sigma', \phi)\}_{n=1}^{\infty}$ is uniformly bounded on $\mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$. So by Lebesgue dominated convergence we need only to prove the pointwise convergence: For almost every $(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$

$$W_B(r_n, r'_n, r'_{*n}, \sigma, \sigma', \phi) \to W_B(r, r', r'_*, \sigma, \sigma', \phi) \quad (n \to \infty). \tag{3.9}$$

The proof is carried out in different cases:

(1) $r' = r'_* = 0$;	(2) $r = r' = 0, r'_* > 0$;
(3) $r = r'_* = 0, r' > 0$;	(4) $r' = 0, r \geqslant r'_* > 0$;
(5) $r'_* = 0, r > 0, r' > 0$;	(6) $r' = 0, r'_* > r > 0$;
(7) $r = 0, r' > 0, r'_* > 0$;	(8) $r > 0, r' > 0, r'_* > 0$.

Let $X_n, Y_n, \cos\theta_n, \sin\theta_n$ and ξ_n be those corresponding to $(r_n, r'_n, r'_{*n}, \sigma, \sigma', \phi)$. By definition of $W_B(r, r', r'_*, \sigma, \sigma', \phi)$ it is easily seen for Case (1)–Case (5) that $W_B(r, r', r'_*, \sigma, \sigma', \phi) = 0$ for all $(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$. [Note that for Case (4) we have $\xi = \sigma$, so $|\langle \xi, \sigma\rangle| = 1$ which implies that $\mathbf{1}_{\{r'_* > r|\langle \xi, \sigma\rangle|\}} = \mathbf{1}_{\{r'_* > r\}} = 0$.]

For Case (1) we have $r'_n \to 0$ and $r'_{*n} \to 0$; for Case (2), $X_n \to 0$, $Y_n \to r'_* > 0$ so that $\cos\theta_n \to 0$; for Case (3), $X_n \to r' > 0$, $Y_n \to 0$ so that $\sin\theta_n \to 0$; and for Case (4) we have $X_n \to r > 0$, $\xi_n \to \sigma$, $|\langle \sigma, \xi_n\rangle| \to 1$ so that $Y_n \to \sqrt{({r'_*}^2 - r^2)^+} = 0$, and so $\sin\theta_n \to 0$. Therefore applying the estimate (3.7) and $B(V, 0) = 0$ we obtain for Case (1)–Case (4) that $W_B(r_n, r'_n, r'_{*n}, \sigma, \sigma', \phi) \to 0$ as $n \to \infty$ for all $(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$. Let

$$\Omega = \{(\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi] \mid |\langle \sigma, \sigma'\rangle| < 1, \phi \neq \pi\}.$$

It is not hard to check that for any $(\sigma, \sigma', \phi) \in \Omega$, $Y = 0$ if and only if $r'_* \leqslant r$ and $r' = 0$, or $r'_* = r = 0$. This property will be used in dealing with Case (8). Now for Case (5) we consider the following $d\sigma d\sigma' d\phi$ – null set:

$$Z_{(r,r')} = \left\{ (\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi] \mid \langle \sigma, \sigma' \rangle = \frac{r}{r'} \right\}.$$

For all $(\sigma, \sigma', \phi) \in \Omega \setminus Z_{(r,r')}$, we have $|\langle \sigma, \xi \rangle| > 0$ and $|\langle \sigma, \xi_n \rangle| \to |\langle \sigma, \xi \rangle|$, $r'_{*n} - r_n |\langle \sigma, \xi_n \rangle| \to -r|\langle \sigma, \xi \rangle| < 0$. This implies that $\mathbf{1}_{\{r'_{*n} > r_n |\langle \sigma, \xi_n \rangle|\}} \to 0$. Therefore $W_B(r_n, r'_n, r'_{*n}, \sigma, \sigma', \phi) \to 0$ as $n \to \infty$ for all $(\sigma, \sigma', \phi) \in \Omega \setminus Z_{(r,r')}$.

Now we deal with the rest of cases. For Case (6), we have $\xi = \sigma$, $X_n \to X = r > 0$, $\xi_n \to \sigma = \xi$, $\langle \sigma, \xi_n \rangle \to 1 = \langle \sigma, \xi \rangle$, $r'_{*n} - r_n |\langle \sigma, \xi_n \rangle| \to r'_* - r > 0$, $Y_n \to \sqrt{{r'_*}^2 - r^2} = Y > 0$, and $\mathbf{1}_{\{r'_{*n} > r_n |\langle \sigma, \xi_n \rangle|\}} \to 1 = \mathbf{1}_{\{r'_* > r|\langle \sigma, \xi \rangle|\}}$. Since the denominators $X > 0, Y > 0$ and $Xr'_* + Yr' = rr'_* > 0$, this implies that the pointwise convergence (3.9) holds for all $(\sigma, \sigma', \phi) \in \Omega$. For Case (7), we have $X_n \to X = r' > 0$, $Y_n \to Y = r'_* > 0$, $X_n r'_{*n} + Y_n r'_n \to Xr'_* + Yr' = 2r'r'_* > 0$, $r'_{*n} - r_n |\langle \sigma, \xi_n \rangle| \to r'_* > 0$ so that $\mathbf{1}_{\{r'_{*n} > r_n |\langle \sigma, \xi_n \rangle|\}} \to 1 = \mathbf{1}_{\{r'_* > r|\langle \sigma, \xi \rangle|\}}$. Thus (3.9) holds for all $(\sigma, \sigma', \phi) \in \Omega$. Finally for Case (8), we consider the following set:

$$Z_{(r,r',r'_*)} = \{ (\sigma, \sigma', \phi) \in \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi] \mid r'_* = r|\langle \sigma, \xi \rangle| \}$$

Note that the equality $r'_* = r|\langle \sigma, \xi \rangle|$ is equivalent to

$$r^2 {r'}^2 \langle \sigma, \sigma' \rangle^2 - 2rr'(r^2 - {r'_*}^2) \langle \sigma, \sigma' \rangle + r^4 - {r'_*}^2 (r^2 + {r'}^2) = 0.$$

Since $r, r' > 0$, the set $Z_{(r,r',r'_*)}$ is a $d\phi d\sigma d\sigma'$-null set. For all $(\sigma, \sigma', \phi) \in \Omega \setminus Z_{(r,r',r'_*)}$, we have $X_n \to X > 0$, $Y_n \to Y > 0$, $X_n r'_{*n} + Y_n r'_n \to Xr'_* + Yr' > 0$, and $r'_{*n} - r_n |\langle \sigma, \xi_n \rangle| \to r'_* - r|\langle \sigma, \xi \rangle| \neq 0$ which implies that $\mathbf{1}_{\{r'_{*n} > r_n |\langle \sigma, \xi_n \rangle|\}} \to \mathbf{1}_{\{r'_* > r|\langle \sigma, \xi \rangle|\}}$ as $n \to \infty$. Therefore (3.9) holds for all $(\sigma, \sigma', \phi) \in \Omega \setminus Z_{(r,r',r'_*)}$ and the proof is completed.

Lemma 3. Let B_n be given in (2.1). Then for any $0 < R < \infty$ and any $\varphi \in C_b^2(\mathbf{R}_+)$,

$$\lim_{n \to \infty} \sup_{(r,r',r'_*) \in [0, R]^3} |W_B(r, r', r'_*) - W_{B_n}(r, r', r'_*)| = 0, \tag{3.10}$$

$$\lim_{n \to \infty} \sup_{(r,r_*) \in [0, R]^2} |J_B[\varphi](r, r_*) - J_{B_n}[\varphi](r, r_*)| = 0, \tag{3.11}$$

$$\lim_{n \to \infty} \sup_{(r,r',r'_*) \in [0, R]^3} |K_B[\varphi](r, r', r'_*) - K_{B_n}[\varphi](r, r', r'_*)| = 0. \tag{3.12}$$

Proof. Since $B_n \leqslant B_{n+1} \leqslant B$ and the factor $(r'^2 - r^2)(r_*'^2 - r^2)$ is independent of (σ, σ', ϕ), it follows from the definition of W_B that the sequence $\{|W_B - W_{B_n}|\}_{n=1}^{\infty}$ is monotone: For all $(r, r', r_*') \in \mathbf{R}_+^3$

$$
\begin{aligned}
&|W_B(r, r', r_*') - W_{B_{n+1}}(r, r', r_*')| = \frac{2}{(4\pi)^3} \iint_{\mathbf{S}^2 \times \mathbf{S}^2} d\sigma d\sigma' \\
&\times \int_0^{2\pi} |W_B(r, r', r_*', \sigma, \sigma', \phi) - W_{B_{n+1}}(r, r', r_*', \sigma, \sigma', \phi)| d\phi \\
&\leqslant |W_B(r, r', r_*') - W_{B_n}(r, r', r_*')|, \qquad n = 1, 2, 3, \dots .
\end{aligned}
$$

Also we have for all $(r, r', r_*', \sigma, \sigma', \phi) \in \mathbf{R}_+^3 \times \mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$

$$
|W_B(r, r', r_*', \sigma, \sigma', \phi) - W_{B_n}(r, r', r_*', \sigma, \sigma', \phi)| \to 0 \quad \text{as} \quad n \to \infty.
$$

Since for any fixed (r, r', r_*') the sequence $\{W_{B_n}(r, r', r_*', \sigma, \sigma', \phi)\}_{n=1}^{\infty}$ is bounded on $\mathbf{S}^2 \times \mathbf{S}^2 \times [0, 2\pi]$ (see the estimate (3.7)), it follows from Lebesgue dominated convergence that

$$
|W_B(r, r', r_*') - W_{B_n}(r, r', r_*')| \to 0 \quad (n \to \infty) \quad \forall\, (r, r', r_*') \in \mathbf{R}_+^3.
$$

On the other hand, by Lemma 2, $|W_B(r, r', r_*') - W_{B_n}(r, r', r_*')|$ are continuous on $\mathbf{R}_+^3$. Therefore the uniform convergence (3.10) follows from Dini's monotone convergence theorem. Using the Dini's monotone convergence again we have (with $B = B(|r\sigma - r_*\sigma_*|, \cos\theta)$)

$$
\begin{aligned}
&\sup_{(r, r_*) \in [0, R]^2} |J_B[\varphi](r, r_*) - J_{B_n}[\varphi](r, r_*)| \\
&\leqslant \|\varphi\|_{L^\infty} \sup_{(r, r_*) \in [0, R]^2} \iint_{\mathbf{S}^2 \times \mathbf{S}^2} d\sigma d\sigma_* \int_0^{\pi/2} |B - B_n| \sin\theta d\theta \\
&\to 0 \qquad (n \to \infty).
\end{aligned}
$$

This proves (3.11). The convergence (3.12) is obvious because of the following estimate

$$
|K_B[\varphi](r, r', r_*') - K_{B_n}[\varphi](r, r', r_*')| \leqslant C_\varphi |W_B(r, r', r_*') - W_{B_n}(r, r', r_*')|
$$

where $C_\varphi = 3\|D^2\varphi\|_{L^\infty}$. ∎

By definitions of $J_B[\varphi]$ and $K_B[\varphi]$ and Lemma 2, it is easily seen that the functions $(r, r_*) \mapsto J_B[\varphi](r, r_*)$ and $(r, r', r'_*) \mapsto K_B[\varphi](r, r', r'_*)$ are continuous on $\mathbf{R}^2_+$ and on $\mathbf{R}^3_+$ respectively. Moreover because $0 \leqslant B_n \leqslant B$ we have for all $\varphi \in C^2_b(\mathbf{R}_+)$ and all $n \geqslant 1$,

$$|J_{B_n}[\varphi](r, r_*)|,\ |J_B[\varphi](r, r_*)| \leqslant C\|\varphi\|_{L^\infty}(1 + r + r_*), \tag{3.13}$$

$$|K_{B_n}[\varphi](r, r', r'_*)|,\ |K_B[\varphi](r, r', r'_*)| \leqslant 3\|D^2\varphi\|_{L^\infty}|W_B(r, r', r'_*)|, \tag{3.14}$$

$$|K_{B_n}[\varphi](r, r', r'_*)|,\ |K_B[\varphi](r, r', r'_*)| \leqslant C\|D^2\varphi\|_{L^\infty}(r' + r'_*)(1 + r' + r'_*). \tag{3.15}$$

Here $C = C_0 \sup_{V \geqslant 0,\, \tau \in [0,1]} \frac{B(V,\tau)}{1+V}$, C_0 is an absolute constant.

Let $\mathcal{B}^+(\mathbf{R}_+)$ be the class of all finite positive Borel measures on $\mathbf{R}_+$. Here as usual the "finite" means that the total variation of a positive measure μ is finite, i.e., $\|\mu\| := \int_{\mathbf{R}_+} d\mu(r) < \infty$. For $s \geqslant 0$, let

$$\mathcal{B}^+_s(\mathbf{R}_+) = \{\mu \in \mathcal{B}^+(\mathbf{R}_+) \mid \|\mu\|_s := \int_{\mathbf{R}^3} (1 + r^s) d\mu(r) < \infty\}.$$

Lemma 4. Given $s \geqslant 0$. Let $\mu_n, \mu \in \mathcal{B}^+_s(\mathbf{R}_+)$ satisfy $\sup_{n \geqslant 1} \|\mu_n\|_s < \infty$ and

$$\lim_{n\to\infty} \int_{\mathbf{R}_+} \psi(r) d\mu_n(r) = \int_{\mathbf{R}_+} \psi(r) d\mu(r) \qquad \forall \psi \in C^\infty_c(\mathbf{R}_+).$$

For any integer $k \geqslant 1$, let $\Psi \in C(\mathbf{R}^k_+)$ satisfy

$$\frac{\Psi(r_1, r_2, \cdots, r_k)}{\prod_{j=1}^k (1 + r_j^s)} \to 0 \quad \text{as} \quad r_1^2 + r_2^2 + \cdots + r_k^2 \to \infty.$$

Then

$$\lim_{n\to\infty} \int_{\mathbf{R}^k_+} \Psi(r_1, r_2, \cdots, r_k) d\mu_n(r_1) d\mu_n(r_2) \cdots d\mu_n(r_k)$$
$$= \int_{\mathbf{R}^k_+} \Psi(r_1, r_2, \cdots, r_k) d\mu(r_1) d\mu(r_2) \cdots d\mu(r_k).$$

In general, if $\{\Psi_n\}_{n=1}^\infty \subset C(\mathbf{R}^k_+)$ satisfies

$$\sup_{n \geqslant 1} \frac{|\Psi_n(r_1, r_2, \cdots, r_k)|}{\prod_{j=1}^k (1 + r_j^s)} \to 0 \qquad \text{as} \quad r_1^2 + r_2^2 + \cdots + r_k^2 \to \infty$$

and for any $R > 0$,

$$\sup_{(r_1,r_2,\cdots,r_k)\in[0,R]^k} |\Psi_n(r_1,r_2,\cdots,r_k)-\Psi(r_1,r_2,\cdots,r_k)|\to 0(n\to\infty),$$

then

$$\lim_{n\to\infty}\int_{\mathbf{R}_+^k}\Psi_n(r_1,r_2,\cdots,r_k)d\mu_n(r_1)d\mu_n(r_2)\cdots d\mu_n(r_k)$$
$$=\int_{\mathbf{R}_+^k}\Psi(r_1,r_2,\cdots,r_k)d\mu(r_1)d\mu(r_2)\cdots d\mu(r_k).$$

Proof. We need only to prove the general case. Let $M=\sup_{n\geqslant 1}\|\mu_n\|_s$. It is easily seen that $\|\mu\|_s\leqslant M$. Let $\mathbf{r}=(r_1,r_2,\cdots,r_k)$,

$$d\nu_n(\mathbf{r})=d\mu_n(r_1)d\mu_n(r_2)\cdots d\mu_n(r_k),\quad d\nu(\mathbf{r})=d\mu(r_1)d\mu(r_2)\cdots d\mu(r_k).$$

Let Ψ_n,Ψ be the functions given in the lemma. Then for any $\epsilon>0$, there exist $R=R_\epsilon>0, n_\epsilon\geqslant 1$ such that

$$|\Psi_n(\mathbf{r})|,\ |\Psi(\mathbf{r})|<\epsilon\prod_{j=1}^k(1+r_j^s),\qquad \forall|\mathbf{r}|\geqslant R,\quad \forall n\geqslant 1,$$
$$|\Psi_n(\mathbf{r})-\Psi(\mathbf{r})|<\epsilon,\qquad \forall\mathbf{r}\in[0,R+2]^k,\quad \forall n\geqslant n_\epsilon.$$

On the other hand, by Weierstrass' approximation theorem, there exists a k-variables polynomial P such that $|\Psi(\mathbf{r})-P(\mathbf{r})|<\epsilon$ for all $\mathbf{r}\in[0,R+2]^k$. Choose a function $\chi_R\in C_c^\infty(\mathbf{R})$ satisfying $0\leqslant\chi_R(r)\leqslant 1$ on $\mathbf{R}$ and $\chi_R(r)=1$ for $r\in[0,R]$ and $\chi_R(r)=0$ for $r\in[R+2,\infty)$. If we write $P(\mathbf{r})=\sum_{i=1}^N\prod_{j=1}^k P_{i,j}(r_j)$ with $P_{i,j}(r)$ the one-variable polynomials, then

$$P(\mathbf{r})\prod_{j=1}^k\chi_R(r_j)=\sum_{i=1}^N\prod_{j=1}^k\psi_{i,j}(r_j)$$

where $\psi_{i,j}(r)=P_{i,j}(r)\chi_R(r)$. Consider the following decomposition

$$\int_{\mathbf{R}_+^k}\Psi_n(\mathbf{r})d\nu_n(\mathbf{r})-\int_{\mathbf{R}_+^k}\Psi(\mathbf{r})d\nu(\mathbf{r})$$
$$=\int_{\mathbf{R}_+^k}\Psi_n\left[1-\prod_{j=1}^k\chi_R(r_j)\right]d\nu_n(\mathbf{r})$$
$$+\int_{\mathbf{R}_+^k}(\Psi_n-\Psi)\prod_{j=1}^k\chi_R(r_j)d\nu_n(\mathbf{r})+\int_{\mathbf{R}_+^k}(\Psi-P)\prod_{j=1}^k\chi_R(r_j)d\nu_n(\mathbf{r})$$
$$+\left[\int_{\mathbf{R}_+^k}\sum_{i=1}^N\prod_{j=1}^k\psi_{i,j}(r_j)d\nu_n(\mathbf{r})-\int_{\mathbf{R}_+^k}\sum_{i=1}^N\prod_{j=1}^k\psi_{i,j}(r_j)d\nu(\mathbf{r})\right]$$

$$+\int_{\mathbf{R}_+^k}(P-\Psi)\prod_{j=1}^{k}\chi_R(r_j)d\nu(\mathbf{r})+\int_{\mathbf{R}_+^k}\Psi\left[\prod_{j=1}^{k}\chi_R(r_j)-1\right]d\nu(\mathbf{r})$$
$$:=I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}+I_5+I_6.$$

Since $1-\prod_{j=1}^{k}\chi_R(r_j)=0$ for all $\mathbf{r}\in[0,R]^k$ and $\prod_{j=1}^{k}\chi_R(r_j)=0$ for all $\mathbf{r}\notin[0,R+2]^k$, it follows that for all $n\geqslant n_\epsilon$

$$|I_{n,1}|\leqslant\int_{\mathbf{R}_+^k\setminus[0,R]^k}|\Psi_n|d\nu_n(\mathbf{r})\leqslant\epsilon\int_{\mathbf{R}_+^k}\prod_{j=1}^{k}(1+r_j^s)d\nu_n(\mathbf{r})\leqslant\epsilon M^k,$$
$$|I_{n,2}|\leqslant\int_{[0,R+2]^k}|\Psi_n-\Psi|d\nu_n(\mathbf{r})\leqslant\epsilon M^k,$$
$$|I_{n,3}|\leqslant\int_{[0,R+2]^k}|\Psi-P|d\nu_n(\mathbf{r})\leqslant\epsilon M^k.$$

Similarly $|I_5|\leqslant\epsilon M^k$, $|I_6|\leqslant\epsilon M^k$. For $I_{n,4}$, since $\psi_{i,j}\in C_c^\infty(\mathbf{R}_+)$, it follows from Fubini theorem and the assumption of the lemma that

$$I_{n,4}=\sum_{i=1}^{N}\prod_{j=1}^{k}\int_{\mathbf{R}_+}\psi_{i,j}(r)d\mu_n(r)-\sum_{i=1}^{N}\prod_{j=1}^{k}\int_{\mathbf{R}_+}\psi_{i,j}(r)d\mu(r)\to 0\qquad\text{as}\quad n\to\infty.$$

Therefore

$$\limsup_{n\to\infty}\left|\int_{\mathbf{R}_+^k}\Psi_n(\mathbf{r})d\nu_n(\mathbf{r})-\int_{\mathbf{R}_+^k}\Psi(\mathbf{r})d\nu(\mathbf{r})\right|\leqslant 5M^k\epsilon\qquad\forall\epsilon>0.$$

This proves the lemma by letting$\epsilon\to 0+$. ∎

Corollary of Lemma 4. Given $s\geqslant 0$. Let $\{\mu_t\}_{t\geqslant 0}\subset\mathcal{B}_s^+(\mathbf{R}_+)$ satisfy

(1) $\sup_{t\in[0,T]}\|\mu_t\|_s<\infty\qquad\forall\, 0<T<\infty$;

(2) for any $\psi\in C_c^\infty(\mathbf{R}_+)$, the function $t\mapsto\int_{\mathbf{R}_+}\psi(r)d\mu_t(r)$ is continuous on $[0,\infty)$.

Then for any integer $k\geqslant 1$ and any $\Psi\in C(\mathbf{R}_+^k\times[0,\infty))$ satisfying

$$\sup_{t\geqslant 0}\frac{|\Psi(r_1,r_2,\cdots,r_k,t)|}{\prod_{j=1}^{k}(1+r_j^s)}\to 0\qquad\text{as}\quad r_1^2+r_2^2+\cdots r_k^2\to\infty,$$

the function

$$t \mapsto \int_{\mathbf{R}_+^k} \Psi(r_1, r_2, \cdots, r_k, t) d\mu_t(r_1) d\mu_t(r_2) \cdots d\mu_t(r_k)$$

is also continuous on $[0, \infty)$ ∎.

Lemma 5. Given $s > 0, M > 0$. Let $\psi \in C_b(\mathbf{R}_+)$. Then for any $\epsilon > 0$, there exists $\varphi_\epsilon \in C_b^k(\mathbf{R}_+)$ (for all $k \geqslant 1$) such that

$$\int_{\mathbf{R}_+} |\varphi_\epsilon(r^2) - \psi(r)| d\mu < \epsilon, \qquad \forall \mu \in \mathcal{B}_s^+(\mathbf{R}_+) \text{ s.t. } \|\mu\|_s \leqslant M.$$

Proof. Given any $\epsilon > 0$. Choose $0 < R < \infty$ such that $\|\psi\|_{L^\infty} M R^{-s} < \epsilon/4$ and then choose $0 < \delta = \delta(\epsilon) < R$ such that

$$|\psi(r) - \psi(r')| < \epsilon/(4M) \qquad \forall r, r' \in [0, 2R] \text{ s.t. } |r - r'| \leqslant 2\delta. \quad (3.16)$$

Let $\psi_\epsilon(r)$ be defined by $\psi_\epsilon(r) = \psi(0)$ for $0 \leqslant r \leqslant 2\delta$, and $\psi_\epsilon(r) = \psi(r)$ for all $r \geqslant 2\delta$. Then $\|\psi_\epsilon - \psi\|_{L^\infty} \leqslant \epsilon$, and $\|\psi_\epsilon\|_{L^\infty} \leqslant \|\psi\|_{L^\infty}$. Let $0 \leqslant \alpha \in C_c^\infty(\mathbf{R})$ with supp $\alpha = [-1, 1]$ and $\int_{-1}^1 \alpha(t) dt = 1$. Let $\alpha_\delta(t) = \alpha(t/\delta)/\delta$,

$$\Psi_\epsilon(r) = \int_{-\infty}^{\infty} \psi_\epsilon(|t|) \alpha_\delta(r - t) dt = \int_{-1}^{1} \psi_\epsilon(|r - \delta t|) \alpha(t) dt. \qquad (3.17)$$

Then $\Psi_\epsilon \in C_b^k(\mathbf{R}_+)$ (for all $k \geqslant 1$) and $\|\Psi_\epsilon\|_{L^\infty} \leqslant \|\psi\|_{L^\infty}$. Also, for all $r \in [0, R]$ and all $t \in [-1, 1]$ we have $|r - \delta t| < 2R$, $||r - \delta t| - r| \leqslant \delta$, and so by (3.16) and (3.17)

$$\sup_{r \in [0, R]} |\Psi_\epsilon(r) - \psi(r)| \leqslant \|\psi_\epsilon - \psi\|_{L^\infty} + \epsilon/(4M) \leqslant \epsilon/(2M).$$

Moreover we note that if $0 \leqslant r \leqslant \delta$, then $|r - \delta t| \leqslant 2\delta$ for all $t \in [-1, 1]$, and so $\psi_\epsilon(|r - \delta t|) = \psi(0)$. Therefore $\Psi_\epsilon(r) = \psi(0)$ for all $r \in [0, \delta]$. Thus $\frac{d}{dr}\Psi_\epsilon(r) = 0$ for all $r \in [0, \delta]$. If we define $\varphi_\epsilon(r) = \Psi_\epsilon(\sqrt{r})$, then it is obvious that $\varphi_\epsilon \in C_b^k(\mathbf{R}_+)$ (for all $k \geqslant 1$) and by the choice of R we have, for any $\mu \in \mathcal{B}_s^+(\mathbf{R}_+)$ satisfying $\|\mu\|_s \leqslant M$,

$$\begin{aligned}
&\int_{\mathbf{R}_+} |\varphi_\epsilon(r^2) - \psi(r)| d\mu(r) \\
&= \int_{\mathbf{R}_+} |\Psi_\epsilon(r) - \psi(r)| d\mu(r) \\
&\leqslant \int_{[0,\, R]} |\Psi_\epsilon(r) - \psi(r)| d\mu(r) + 2\|\psi\|_{L^\infty} \int_{[R,\, \infty)} d\mu(r) \\
&\leqslant \epsilon/2 + 2\|\psi\|_{L^\infty} M R^{-s} < \epsilon. \qquad ∎
\end{aligned}$$

Lemma 6. Let F be a positive Borel measure on $\mathbf{R}_+$ with $0 < \int_{\mathbf{R}_+}(1+r^s)dF(r) < \infty$ for some $s \geqslant 2$. Then there exist isotropic functions $0 < f^n \in L^1_2(\mathbf{R}^3) \cap C(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} f^n(|v|)\psi(|v|)dv \to \int_{\mathbf{R}_+} \psi(r)dF(r) \qquad (n \to \infty)$$

for all $\psi \in C(\mathbf{R}_+)$ satisfying $\sup_{r \geqslant 0} |\psi(r)|/(1+r^s) < \infty$. Moreover, if $\int_{\mathbf{R}_+} r^2 dF(r) > 0$, then for all $n \geqslant 1$

$$\int_{\mathbf{R}^3} f^n(|v|)dv = \int_{\mathbf{R}_+} dF(r), \quad \int_{\mathbf{R}^3} f^n(|v|)|v|^2 dv = \int_{\mathbf{R}_+} r^2 dF(r). \quad (3.18)$$

Proof. By Riesz representation theorem, there exists a unique finite positive Borel measure μ on $\mathbf{R}^3$ such that for all $\psi \in C_0(\mathbf{R}^3)$

$$\int_{\mathbf{R}^3} \psi(v)d\mu(v) = \frac{1}{4\pi}\int_{\mathbf{R}_+}\left(\int_{\mathbf{S}^2}\psi(r\omega)d\omega\right)dF(r). \quad (3.19)$$

It is easily proved using monotone convergence that

$$\int_{\mathbf{R}^3}(1+|v|^s)d\mu(v) = \int_{\mathbf{R}_+}(1+r^s)dF(r) < \infty$$

and so Eq. (3.19) holds for all $\psi \in C(\mathbf{R}^3)$ satisfying $\sup_{v\in\mathbf{R}^3}|\psi(v)|/(1+|v|^s) < \infty$.

Let $T = \frac{1}{3}\int_{\mathbf{R}_+} r^2 dF(r)/\int_{\mathbf{R}^3} dF(r)$ for $\int_{\mathbf{R}_+} r^2 dF(r) > 0$; $T = 1$ for $\int_{\mathbf{R}_+} r^2 dF(r) = 0$, and let

$$M(|v|) = \frac{1}{(2\pi T)^{3/2}}\exp(-|v|^2/2T).$$

Let f^n be given by the Mehler transform (thanks to Carlen's suggestion):

$$f^n(v) = e^{3n}\int_{\mathbf{R}^3} M\left(e^n|v - \sqrt{1-e^{-2n}}\,v_*|\right)d\mu(v_*).$$

By definition of the measure μ, the functions f^n are strictly positive and continuous on $\mathbf{R}^3$. Moreover, it is easily verified that f^n are isotropic functions. So we can write $f^n(v) = f^n(|v|)$. By Fubini theorem and changing variable $z = e^n(v - \sqrt{1-e^{-2n}}\, v_*)$, we compute

$$\int_{\mathbf{R}^3} f^n(|v|)\psi(|v|)dv = \int_{\mathbf{R}_+} I_n[\psi](v_*)d\mu(v_*) \tag{3.20}$$

for all $\psi \in C(\mathbf{R}_+)$ satisfying $\sup_{r\in\mathbf{R}_+} |\psi(r)|/(1+r^s) < \infty$, where

$$I_n[\psi](v_*) = \int_{\mathbf{R}^3} \psi(|e^{-n} z + \sqrt{1-e^{-2n}}\, v_*|)M(|z|)dz. \tag{3.21}$$

Since

$$\begin{aligned} &|\psi(|e^{-n} z + \sqrt{1-e^{-2n}}\, v_*|)| \\ &\leqslant C(1+|e^{-n} z + \sqrt{1-e^{-2n}}\, v_*|^s) \leqslant C(1+|z|^s)(1+|v_*|^s), \end{aligned}$$

it follows from dominated convergence that

$$I_n[\psi](v_*) \to \psi(|v_*|) \qquad (n\to\infty) \qquad \forall\, v_* \in \mathbf{R}^3.$$

Also we have, for some constant C, $|I_n[\psi](v_*)| \leqslant C(1+|v_*|^s)$. Thus applying dominated convergence again we obtain

$$\begin{aligned} &\int_{\mathbf{R}^3} f^n(|v|)\psi(|v|)dv = \int_{\mathbf{R}^3} I_n[\psi](v_*)d\mu(v_*) \\ &\to \int_{\mathbf{R}^3} \psi(|v_*|)d\mu(v_*) = \int_{\mathbf{R}_+} \psi(r)dF(r) \quad (n\to\infty). \end{aligned}$$

Finally, if $\int_{\mathbf{R}_+} r^2 dF(r) > 0$, then applying (3.20) and (3.21) to $\psi(|v|) = 1, |v|^2$ and recalling the choice of T and $M(|v|)$ we obtain (3.18). ∎

4. WEAK STABILITY AND EXISTENCE OF SOLUTIONS

Theorem 1. (**Weak Stability**). Let B, B_n be collision kernels satisfying the conditions (i)–(ii) and either $B_n \equiv B$ ($\forall n \geqslant 1$) or B_n be the cutoff of B given by (2.1) ($\forall n \geqslant 1$). Let F_0, F_0^n be positive Borel measures on $\mathbf{R}_+$ satisfying

$$\int_{\mathbf{R}_+} (1+r^2)dF_0(r) < \infty, \qquad \sup_{n\geqslant 1} \int_{\mathbf{R}_+} (1+r^2)dF_0^n(r) < \infty,$$

and

$$\lim_{n\to\infty}\int_{\mathbf{R}_+}\psi(r)dF_0^n(r)=\int_{\mathbf{R}_+}\psi(r)dF_0(r)\qquad \forall\psi\in C_b(\mathbf{R}_+).$$

Let F_t^n be conservative isotropic distributional solutions of Eq. (BBE) with the kernel B_n and $F_t^n|_{t=0}=F_0^n$ ($\forall n\geqslant 1$). Then there exist a subsequence $\{F_t^{n_j}\}_{j=1}^{\infty}$ of $\{F_t^n\}_{n=1}^{\infty}$ and an isotropic distributional solution F_t of Eq. (BBE) with the kernel B satisfying $F_t|_{t=0}=F_0$ such that

$$\lim_{j\to\infty}\int_{\mathbf{R}_+}\psi(r)dF_t^{n_j}(r)=\int_{\mathbf{R}_+}\psi(r)dF_t(r)\quad \forall t\geqslant 0,\quad \forall\psi\in C_b(\mathbf{R}_+)\quad (4.1)$$

(therefore F_t conserves the mass) and

$$\int_{\mathbf{R}_+}r^2dF_t(r)\leqslant \liminf_{j\to\infty}\int_{\mathbf{R}^3}r^2dF_0^{n_j}(r)\quad \forall t\geqslant 0. \quad (4.2)$$

Furthermore if

$$\lim_{n\to\infty}\int_{\mathbf{R}_+}r^2dF_0^n(r)=\int_{\mathbf{R}_+}r^2dF_0(r), \quad (4.3)$$

then the solution F_t also conserves the energy:

$$\int_{\mathbf{R}_+}r^2dF_t(r)=\int_{\mathbf{R}^3}r^2dF_0(r)\qquad \forall t\geqslant 0.$$

Proof. By conservation of the mass and energy, we have $\sup_{n\geqslant 1,t\geqslant 0}\|F_t^n\|_2=M<\infty$. Here and below $M=\sup_{n\geqslant 1}\|F_0^n\|_2$. Let

$$\Lambda_n[\psi](t)=\int_{\mathbf{R}_+}\psi(r)dF_t^n(r),\qquad t\in[0,\infty),\quad \psi\in C_c(\mathbf{R}_+). \quad (4.4)$$

Then $|\Lambda_n[\psi](t)|\leqslant M\|\psi\|_{L^\infty}$ for all $n\geqslant 1, t\geqslant 0$. Since $C_c(\mathbf{R}_+)$ is separable under the distance $\mathrm{dist}(\psi_1,\psi_2)=\|\psi_1-\psi_2\|_{L^\infty}$, applying the usual diagonal process we obtain a subsequence of positive integers, still denote it as $\{n\}$, such that $\Lambda[\psi](\bar t):=\lim_{n\to\infty}\Lambda_n[\psi](\bar t)$ exists for all $\psi\in C_c(\mathbf{R}_+)$ and all $\bar t\in\mathbf{Q}_+$. Here $\mathbf{Q}_+$ is the set of nonnegative rational numbers. Now we prove that this limit also exists for all real number $t\geqslant 0$. The main step for proving this is to prove the following equi-continuity: For any $\psi \in C_c(\mathbf{R}_+)$,

$$\sup_{n\geqslant 1} |\Lambda_n[\psi](t_1) - \Lambda_n[\psi](t_2)| \to 0 \quad as \quad |t_1 - t_2| \to 0. \tag{4.5}$$

Recall that F_t^n are distributional solutions of Eq. (BBE). We have for any $\varphi \in C_b^2(\mathbf{R}_+)$

$$\begin{aligned}
&\int_{\mathbf{R}_+} \varphi(r^2) dF_t^n(r) \\
&= \int_{\mathbf{R}_+} \varphi(r^2) dF_0^n(r) + \int_0^t ds \iint_{\mathbf{R}_+^2} J_{B_n}[\varphi](r, r_*)\, dF_s^n(r)\, dF_s^n(r_*) \\
&\quad + \varepsilon \int_0^t ds \iiint_{\mathbf{R}_+^3} K_{B_n}[\varphi](r, r', r'_*)\, dF_s^n(r)\, dF_s^n(r')\, dF_s^n(r'_*), \quad t \geqslant 0.
\end{aligned}$$

By inequalities (3.13) and (3.15), this implies that $\forall\, t_1, t_2 \geqslant 0$

$$\sup_{n\geqslant 1} \left| \int_{\mathbf{R}_+} \varphi(r^2) dF_{t_1}^n(r) - \int_{\mathbf{R}_+} \varphi(r^2) dF_{t_2}^n(r) \right| \leqslant C \|\varphi\|_{2,\infty} |t_1 - t_2| \tag{4.6}$$

where $\|\varphi\|_{2,\infty} = \|\varphi\|_{L^\infty} + \|D^2\varphi\|_{L^\infty}$, C depends only on B and M. For any $\epsilon > 0$, by Lemma 5 (with $s = 2$), there exists $\varphi_\epsilon \in C_b^2(\mathbf{R}_+)$ such that

$$\sup_{n\geqslant 1, t\geqslant 0} \left| \int_{\mathbf{R}_+} \varphi_\epsilon(r^2) dF_t^n(r) - \int_{\mathbf{R}_+} \psi(r) dF_t^n(r) \right| < \epsilon.$$

This together with (4.6) gives

$$\sup_{n\geqslant 1} |\Lambda_n[\psi](t_1) - \Lambda_n[\psi](t_2)| < 2\epsilon + C\|\varphi_\epsilon\|_{2,\infty} |t_1 - t_2|$$

which implies (4.5). The property (4.5) implies that $|\Lambda[\psi](\bar{t}_1) - \Lambda[\psi](\bar{t}_2)| \to 0$ as $|\bar{t}_1 - \bar{t}_2| \to 0$ for $\bar{t}_1, \bar{t}_2 \in \mathbf{Q}_+$. So $\Lambda[\psi](t) := \lim_{\bar{t}\to t} \Lambda[\psi](\bar{t})$ exists for all $t \in [0, \infty)$ and all $\psi \in C_c(\mathbf{R}_+)$. Therefore $\lim_{n\to\infty} \Lambda_n[\psi](t) = \Lambda[\psi](t)$ holds for all $t \in [0, \infty)$ and all $\psi \in C_c(\mathbf{R}_+)$. Since for every fixed t, $\psi \mapsto \Lambda[\psi](t)$ is a bounded positive linear functional on $C_c(\mathbf{R}_+)$, it follows from Riesz representation theorem that there exists a unique finite positive Borel measure F_t on $\mathbf{R}_+$ such that $\Lambda[\psi](t)=\int_{\mathbf{R}_+}\psi(r)dF_t(r)$, i.e.,

$$\lim_{n\to\infty}\int_{\mathbf{R}_+}\psi(r)dF_t^n(r)=\int_{\mathbf{R}_+}\psi(r)dF_t(r)\quad \forall\,\psi\in C_c(\mathbf{R}_+),\quad \forall\, t\in[0,\infty). \tag{4.7}$$

By the assumption of the theorem, this implies that $F_t|_{t=0}=F_0$. Combining (4.7) with (4.4)–(4.5) one sees that the function $t\mapsto\int_{\mathbf{R}_+}\psi(r)dF_t(r)$ is continuous on $[0,\infty)$ for all $\psi\in C_c(\mathbf{R}_+)$. Since F_t^n are conservative solutions, it follows that F_t conserves the mass and satisfies (4.1) and (4.2).

Now we are going to prove that F_t is a distributional solution to Eq. (BBE) with the kernel B. Recall that for any $\varphi\in C_b^2(\mathbf{R}_+)$, the functions $J_B[\varphi](r,r_*)$, $J_{B_n}[\varphi](r,r_*)$, $K_B[\varphi](r,r',r_*')$, and $K_{B_n}[\varphi](r,r',r_*')$ are continuous on $\mathbf{R}_+^2$ and $\mathbf{R}_+^3$ respectively. And by the inequalities (3.13), (3.14), and Lemma 2 (3.6) we have for all $\varphi\in C_b^2(\mathbf{R}_+)$,

$$\lim_{r^2+r_*^2\to\infty}\frac{|J_B[\varphi](r,r_*)|+\sup_{n\geqslant 1}|J_{B_n}[\varphi](r,r_*)|}{(1+r^2)(1+r_*^2)}=0, \tag{4.8}$$

$$\lim_{r^2+r'^2+r_*'^2\to\infty}\frac{|K_B[\varphi](r,r',r_*')|+\sup_{n\geqslant 1}|K_{B_n}[\varphi](r,r',r_*')|}{(1+r^2)(1+r'^2)(1+r_*'^2)}=0. \tag{4.9}$$

Therefore by (4.7), (4.8), (4.9), Lemma 3, and Lemma 4 (with $s=2$) we obtain that for any $\varphi\in C_b^2(\mathbf{R}_+)$ and any $t\in[0,\infty)$,

$$\lim_{n\to\infty}\int_{\mathbf{R}_+}\varphi(r^2)dF_t^n(r)=\int_{\mathbf{R}_+}\varphi(r^2)dF_t(r),$$

$$\lim_{n\to\infty}\iint_{\mathbf{R}_+^2}J_{B_n}[\varphi](r,r_*)\,dF_t^n(r)\,dF_t^n(r_*)$$
$$=\iint_{\mathbf{R}_+^2}J_B[\varphi](r,r_*)\,dF_t(r)\,dF_t(r_*),$$

$$\lim_{n\to\infty}\iiint_{\mathbf{R}_+^3}K_{B_n}[\varphi](r,r',r_*')\,dF_t^n(r)\,dF_t^n(r')\,dF_t^n(r_*')$$
$$=\iiint_{\mathbf{R}_+^3}K_B[\varphi](r,r',r_*')\,dF_t(r)\,dF_t(r')\,dF_t(r_*').$$

Also we have by (3.13),(3.15), and (4.2) that the collision integrals are bounded: For all $n \geqslant 1$ and all $t \geqslant 0$,

$$\iint_{\mathbf{R}_+^2} |J_B[\varphi](r, r_*)|\, dF_t^n(r)\, dF_t^n(r_*) \leqslant C\|\varphi\|_{L^\infty} M^2,$$
$$\iint_{\mathbf{R}_+^2} |J_B[\varphi](r, r_*)|\, dF_t(r)\, dF_t(r_*) \leqslant C\|\varphi\|_{L^\infty} M^2,$$
$$\iiint_{\mathbf{R}_+^3} |K_{B_n}[\varphi](r, r', r_*')|\, dF_t^n(r)\, dF_t^n(r')\, dF_t^n(r_*') \leqslant C\|D^2\varphi\|_{L^\infty} M^3,$$
$$\iiint_{\mathbf{R}_+^3} |K_B[\varphi](r, r', r_*')|\, dF_t(r)\, dF_t(r')\, dF_t(r_*') \leqslant C\|D^2\varphi\|_{L^\infty} M^3,$$

where C depends only on B. Thus by Lebesgue dominated convergence we obtain for all $t \geqslant 0$

$$\begin{aligned}&\int_{\mathbf{R}_+} \varphi(r^2) dF_t(r)\\ &= \int_{\mathbf{R}_+} \varphi(r^2) dF_0(r) + \int_0^t ds \iint_{\mathbf{R}_+^2} J_B[\varphi](r, r_*)\, dF_s(r)\, dF_s(r_*)\\ &\quad + \varepsilon \int_0^t ds \iiint_{\mathbf{R}_+^3} K_B[\varphi](r, r', r_*')\, dF_s(r)\, dF_s(r')\, dF_s(r_*').\end{aligned}$$

Furthermore, (4.8), (4.9) and the Corollary of Lemma 4 imply that the functions

$$t \mapsto \iint_{\mathbf{R}_+^2} J_B[\varphi](r, r_*)\, dF_t(r)\, dF_t(r_*),$$
$$t \mapsto \iiint_{\mathbf{R}_+^3} K_B[\varphi](r, r', r_*')\, dF_t(r)\, dF_t(r')\, dF_t(r_*')$$

are continuous on $[0, \infty)$. Therefore we have for any $\varphi \in C_b^2(\mathbf{R}_+)$

$$\begin{aligned}&\frac{d}{dt} \int_{\mathbf{R}_+} \varphi(r^2) dF_t(r) = \iint_{\mathbf{R}_+^2} J_B[\varphi](r, r_*)\, dF_t(r)\, dF_t(r_*)\\ &+ \varepsilon \iiint_{\mathbf{R}_+^3} K_B[\varphi](r, r', r_*')\, dF_t(r)\, dF_t(r')\, dF_t(r_*') \quad \forall t \geqslant 0.\end{aligned}$$

Thus F_t is an isotropic distributional solution of Eq. (BBE). Finally, suppose the condition (4.3) is satisfied. Then (4.2) and (4.3) imply

$$\int_{\mathbf{R}_+} r^2 dF_t(r) \leqslant \int_{\mathbf{R}_+} r^2 dF_0(r) \qquad \forall t \geqslant 0. \tag{4.10}$$

On the other hand, because F_t is a solution, the energy $t \mapsto \int_{\mathbf{R}_+^2} r^2 dF_t(r)$ is non-decreasing on $[0, \infty)$ (see Theorem 3 in the next section). This together with (4.10) implies that F_t conserves the energy. The proof of the theorem is completed. ∎

Theorem 2. **(Existence)**. Let B be a collision kernel satisfying the conditions (i)–(ii). Let F_0 be a positive Borel measure on $\mathbf{R}_+$ satisfying $0 < \int_{\mathbf{R}_+} (1 + r^2) dF_0(r) < \infty$. Then Eq. (BBE) has an isotropic distributional solution F_t with $F_t|_{t=0} = F_0$ and F_t conserves the mass and energy.

Proof. For any $n \geqslant 1$, let $f_0^n \in L_2^1(\mathbf{R}^3)$ be the nonnegative isotropic function obtained in Lemma 6 (with $s = 2$) corresponding to the measurer F_0. Let B_n be the cutoff kernel given in Eq. (2.1). By Theorem 3 in ref. 20, the Eq. (BBE) with the kernel B_n has a conservative isotropic solution $f^n(|v|, t)$ in the class $C^1([0, \infty); L_2^1(\mathbf{R}^3))$ satisfying $f^n(|v|, t)|_{t=0} = f_0^n(|v|)$. As in Section 2, let $dF_0^n(r) = 4\pi r^2 f_0^n(r) dr$, $dF_t^n(r) = 4\pi r^2 f^n(r, t) dr$. Then F_t^n are conservative distributional solutions of Eq. (BBE) with the kernels B_n satisfying $F_t^n|_{t=0} = F_0^n$. Also by Lemma 6 we have $\lim_{n\to\infty} \int_{\mathbf{R}_+} r^2 dF_0^n(r) = \int_{\mathbf{R}_+} r^2 dF_0(r)$. Thus the theorem follows from Theorem 1. ∎

Remark 2. For any distributional solution F_t of Eq. (BBE), Lemma 5 (with $s = 2$) implies that the function $t \mapsto \int_{\mathbf{R}_+} \psi(r) dF_t(r)$ is continuous on $[0, \infty)$ for all $\psi \in C_b(\mathbf{R}_+)$. From this property and the Corollary of Lemma 4 (with $s = 2$) one sees that the proof given in Section 2 for the general weak form (2.13) of Eq. (BBE) is rigorous.

5. ENERGY INEQUALITY AND MOMENT PRODUCTION

This section deals with the estimates of moments $\int_{\mathbf{R}_+} r^s dF_t(r)$ $(s \geqslant 2)$. We first prove that the second order moment (i.e., the energy) is non-decreasing for all solutions.

Theorem 3. Let B a collision kernel satisfying the conditions (i)–(ii), and let F_0 be a positive Borel measure on $\mathbf{R}_+$ with $\int_{\mathbf{R}_+} (1 +$

$r^2)dF_0(r) < \infty$. Then for any isotropic distributional solution F_t of Eq. (BBE) with initial datum $F_t|_{t=0} = F_0$, we have

$$\int_{\mathbf{R}_+} r^2 dF_s(r) \leqslant \int_{\mathbf{R}_+} r^2 dF_t(r) \qquad \forall\, 0 \leqslant s < t < \infty. \tag{5.1}$$

Proof. The proof is similar to that given in ref. 19 using suitable truncation of r^2, but this time the truncation should be in $C_b^2(\mathbf{R}_+)$ because of our Definition of distributional solutions. Let $\delta > 0, \varphi_\delta(r) = r/(1+\delta r)$. Then $\varphi_\delta \in C_b^2(\mathbf{R}_+)$ and we compute

$$\Delta\varphi_\delta = \varphi_\delta(|v|^2) + \varphi_\delta(|v_*|^2) - \varphi_\delta(|v'|^2) - \varphi_\delta(|v'_*|^2) = \Phi_\delta^{(+)} - \Phi_\delta^{(-)}$$

where $\Phi_\delta^{(\pm)} \geqslant 0$,

$$\Phi_\delta^{(+)} = \frac{\delta[2+\delta(|v|^2+|v_*|^2)]|v|^2|v_*|^2}{(1+\delta|v|^2)(1+\delta|v_*|^2)(1+\delta|v'|^2)(1+\delta|v'_*|^2)}$$

and $\Phi_\delta^{(-)}$ has the same form as $\Phi_\delta^{(+)}$ by replacing $|v|^2|v_*|^2$ with $|v'|^2|v'_*|^2$. By symmetry of the collision integral, we have

$$\iint_{\mathbf{R}_+^2} J_B[\varphi_\delta](r,r_*)dF_t(r)dF_t(r_*) = \frac{1}{2}\iint_{\mathbf{R}_+^2} \widehat{J}_B[\varphi_\delta](r,r_*)dF_t(r)dF_t(r_*)$$

where

$$\widehat{J}_B[\varphi_\delta](r,r_*) = \frac{1}{(4\pi)^2}\iiint_{\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2} B(r\sigma - r_*\sigma_*,\omega)[-\Delta\varphi_\delta]d\omega d\sigma d\sigma_*.$$

Let

$$I_\delta(r,r_*) = \frac{1}{(4\pi)^2}\iiint_{\mathbf{S}^2\times\mathbf{S}^2\times\mathbf{S}^2} B(r\sigma - r_*\sigma_*,\omega)\Phi_\delta^{(+)}d\omega d\sigma d\sigma_*.$$

Then $\widehat{J}_B[\varphi_\delta](r,r_*) \geqslant -I_\delta(r,r_*)$. A a simple estimate using $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ gives $\Phi_\delta^{(+)} \leqslant |v||v_*| = rr_*$. This together with $B(r\sigma - r_*\sigma_*,\omega) \leqslant C(1+r+r_*)$ implies

$$0 \leqslant I_\delta(r,r_*) \leqslant C(1+r^2)(1+r_*^2). \tag{5.2}$$

Also, since $\|D^2\varphi_\delta\|_{L^\infty} \leqslant 2\delta$, it follows from (2.7), (2.8) and (3.5) that

$$|K_B[\varphi_\delta](r, r', r'_*)| \leqslant C\delta(1+r'^2)(1+r_*'^2).$$

Here the constant $C<\infty$ depends only on B. Thus for any $0 \leqslant s < t < \infty$ we have

$$\int_{\mathbf{R}_+} \varphi_\delta(r^2)dF_t(r) \geqslant \int_{\mathbf{R}_+} \varphi_\delta(r^2)dF_s(r)$$
$$-\int_s^t d\tau \iint_{\mathbf{R}_+^2} I_\delta(r, r_*)dF_\tau(r)dF_\tau(r_*) - C\varepsilon\delta(\sup_{0\leqslant\tau\leqslant t} \|F_\tau\|_2)^3 t. \quad (5.3)$$

It is easily seen that $I_\delta(r, r_*) \to 0$ $(\delta \to 0+)$. This together with (5.2) and dominated convergence implies that

$$\int_s^t d\tau \iint_{\mathbf{R}_+^2} I_\delta(r, r_*)dF_\tau(r)dF_\tau(r_*) \to 0 \qquad (\delta \to 0+).$$

Also by $0 \leqslant \varphi_\delta(r^2) \leqslant r^2$ and $\varphi_\delta(r^2) \to r^2(\delta \to 0+)$ we have

$$\int_{\mathbf{R}_+} \varphi_\delta(r^2)dF_\tau(r) \to \int_{\mathbf{R}_+} r^2 dF_\tau(r) \quad (\delta \to 0+) \quad \forall \tau \in [0, t].$$

Therefore the inequality (5.3) leads to (5.1) by letting $\delta \to 0+$. ∎

Now we are going to establish the moment production estimates of distributional solutions of Eq. (BBE) for the original hard sphere model and for hard potential models with angular cutoff as mentioned in the Introduction.

Theorem 4. Let $B(|v-v_*|, \cos\theta) = b(\cos\theta)|v-v_*|^\gamma$, where $0 < \gamma \leqslant 1$ is constant, $0 \leqslant b \in C([0,1])$ and satisfies $b(0)=0$, $\int_0^1 b(\tau)d\tau > 0$. Let F_0 be a positive Borel measure on $\mathbf{R}_+$ with $0 < M_0 = \int_{\mathbf{R}_+} dF_0(r) < \infty$, $M_2 = \int_{\mathbf{R}^3} r^2 dF_0(r) < \infty$. Then Eq. (BBE) with the kernel B has a conservative distributional solution F_t which satisfies $F_t|_{t=0} = F_0$ and for any $s > 2$

$$\int_{\mathbf{R}_+} r^s dF_t(r) \leqslant C_s \left(1+\frac{1}{t}\right)^{(s-2)/\gamma} \qquad \forall t > 0 \qquad (5.4)$$

where $C_s = C_{B,\varepsilon,s}(M_0, M_2) < \infty$ depends only on M_0, M_2, B, ε and s, and the function $C_{B,\varepsilon,s}(x_1, x_2)$ is continuous on $(x_1, x_2) \in (0, \infty) \times [0, \infty)$.

Proof. The proof is an improvement of that given in ref. 20 where the moment estimate similar to (5.4) was established for approximate solutions $f^n(|v|, t)$ for the kernel B_n given in Eq. (2.1), but the constants in the upper bounds of moment estimates tend to infinity as $n \to \infty$. This is because that the property of $K_{B_n}[\varphi]$ was not considered to control the constants.

Now suppose for any $n \geqslant 1$ the initial datum $f_0^n(|v|)$ is given as in the proof of Theorem 2. By ref. 20 the conservative solution $f^n(|v|, t)$ with initial datum $f_0^n(|v|)$ is unique and belongs to the class $C^1((0, \infty); L_s^1(\mathbf{R}^3))$ for all $s > 2$. Let $F_0^n(r)$, F_t^n be denoted as in the proof of Theorem 2 and let F_t be a conservative distributional solution obtained in the same proof, i.e., F_t is a weak limit of a subsequence of $\{F_t^n\}_{n=1}^{\infty}$ and satisfies $F_t|_{t=0} = F_0$. We shall prove that this solution F_t satisfies the estimate (5.4). For any $F \in \mathcal{B}_s^+(\mathbf{R}_+)$ let

$$M_s(F) = \int_{\mathbf{R}_+} r^s \, dF(r).$$

It suffice to prove that the approximate solutions F_t^n satisfy that for any $s > 2$, there is a positive continuous function $(x_1, x_2) \mapsto C_{B,\varepsilon,s}(x_1, x_2)$ on $(0, \infty) \times [0, \infty)$ which is determined only by B, ε, and s, such that

$$M_s(F_t^n) \leqslant C_{s,n} \, (1 + 1/t)^{(s-2)/\gamma} \qquad \forall t > 0 \tag{5.5}$$

where $C_{s,n} = C_{B,\varepsilon,s}(M_0(F_0^n), M_2(F_0^n))$. In fact if (5.5) holds for all F_t^n, then by taking weak limit together with the continuity

$$C_{B,\varepsilon,s}(M_0(F_0^n), M_2(F_0^n)) \to C_{B,\varepsilon,s}(M_0(F_0), M_2(F_0)) \qquad (n \to \infty)$$

one sees that F_t satisfies (5.4). We note also that if the estimate (5.5) holds for all $s \geqslant 4$, then it holds for all $s > 2$ (thanks to Wennberg's suggestion). In fact by Hölder inequality and (5.5) with $s = 4$ we have for $2 < s < 4$

$$\begin{aligned} M_s(F_t^n) &\leqslant [M_2(F_t^n)]^{(4-s)/2} [M_4(F_t^n)]^{(s-2)/2} \\ &\leqslant [M_2(F_0^n)]^{(4-s)/2} [C_{4,n}]^{(s-2)/2} \, (1 + 1/t)^{(s-2)/\gamma} \qquad \forall t > 0. \end{aligned}$$

Therefore by defining $C_{B,\varepsilon,s}(x_1, x_2) = x_2^{(4-s)/2} [C_{B,\varepsilon,4}(x_1, x_2)]^{(s-2)/2}$ for $2 < s < 4$, the estimate (5.5) holds for all $2 < s < 4$ and therefore for all $s > 2$. Therefore to prove (5.4) we need only to prove (5.5) for $s \geqslant 4$.

As in ref. 20, we use the following version of Povzner-Elmroth inequality (see e.g., ref. 19 and recall that $s \geqslant 4$):

$$|v'|^s + |v'_*|^s - |v|^s - |v_*|^s \leqslant 2^s \left(|v|^{s-\gamma}|v_*|^\gamma + |v|^\gamma |v_*|^{s-\gamma}\right) - 2^{-s}[\kappa(\theta)]^s |v|^s$$

where $\kappa(\theta) = \min\{\cos\theta, 1-\cos\theta\}$, $\theta = \arccos(|v-v_*|^{-1}|\langle v-v_*, \omega\rangle|)$. Write $B_n = B_n(v-v_*, \omega)$. By conservation of mass, we have for all $n \geqslant 1$ and all $t > 0$

$$\begin{aligned}
&\frac{d}{dt}\|F_t^n\|_s \\
&\quad = \frac{d}{dt} M_s(F_t^n) \\
&\quad \leqslant 2^{s-1} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n\, f^n f_*^n \left(|v|^{s-\gamma}|v_*|^\gamma + |v|^\gamma|v_*|^{s-\gamma}\right) d\omega dv_* dv \\
&\quad\quad -2^{-s-1} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n\, f^n f_*^n [\kappa(\theta)]^s |v|^s d\omega dv_* dv \\
&\quad\quad +\varepsilon \iiint_{\mathbf{R}^3_+} K_{B_n}[\varphi](r, r', r'_*) dF_t^n(r) dF_t^n(r') dF_t^n(r'_*) \\
&\quad := 2^{s-1} I_n^{(1)} - 2^{-s-1} I_n^{(2)} + \varepsilon I_n^{(3)}.
\end{aligned} \tag{5.6}$$

Here $\varphi(r) = r^{s/2}$. Let $b_\infty = \max_{\tau\in[0,1]} b(\tau)$,

$$A_s = 4\pi \int_0^{\pi/2} [\kappa(\theta)]^s \min\{\cos^2\theta \sin\theta,\, b(\cos\theta)\} \sin\theta\, d\theta,$$

and let $C_s^{(1)}, C_s^{(2)}, \dots$ denote finite and strictly positive constants that depend only on b_∞, γ, A_s and s. Then following the proof in ref. 20 we have

$$\begin{aligned}
I_n^{(1)} &\leqslant C_s^{(1)} \|F_0^n\|_2 \|F_t^n\|_s, \\
I_n^{(2)} &\geqslant \frac{1}{2} A_s \|F_0^n\|_0 M_{s+\gamma}(F_t^n) - \frac{1}{2} A_s \|F_0^n\|_0 \|F_t^n\|_s.
\end{aligned}$$

Further, using the relation $M_{s+\gamma}(F_t^n) = \|F_t^n\|_{s+\gamma} - \|F_0^n\|_0$ and the inequalities $1 + r^s \leqslant (1+r^2)^{s/2}$, $(1+r^2)^{(s+\gamma)/2} \leqslant 2^{(s+\gamma-2)/2}(1+r^{s+\gamma})$, and applying Hölder inequality, we have

$$M_{s+\gamma}(F_t^n) \geqslant 2^{-(s+\gamma-2)/2} \left(\|F_0^n\|_2\right)^{-\gamma/(s-2)} \left(\|F_t^n\|_s\right)^{1+\gamma/(s-2)} - \|F_0^n\|_0.$$

Since $\|F_0^n\|_0 = \|F_t^n\|_0 \leqslant \|F_t^n\|_s$, this gives

$$\begin{aligned} 2^{s-1} I_n^{(1)} - 2^{-s-1} I_n^{(2)} \leqslant\ & C_s^{(3)} \|F_0^n\|_2 \|F_t^n\|_s \\ & -C_s^{(4)} \|F_0^n\|_0 \left(\|F_0^n\|_2\right)^{-\gamma/(s-2)} \left(\|F_t^n\|_s\right)^{1+\gamma/(s-2)}. \end{aligned} \tag{5.7}$$

For the third term, $I_n^{(3)}$, we compute using the estimate (2.7) for $\varphi(r) = r^{s/2}$ that $|L[D^2\varphi](r, r', r'_*)| \leqslant \frac{3}{4}s(s-2)(r' + r'_*)^{s-4}$ (because $s \geqslant 4$) and so by Eqs. (2.8) and (3.5) we obtain

$$|K_{B_n}[\varphi](r, r', r'_*)| \leqslant 60 s^2 b_\infty (1 + r' + r'_*)^{s-2}$$

which gives

$$I_n^{(3)} \leqslant C_s^{(5)} \left(\|F_0^n\|_0\right)^2 \|F_t^n\|_s.$$

Combining this with (5.6) and (5.7) we see that $\|F_t^n\|_s$ satisfies the following differential inequality

$$\begin{aligned} \frac{d}{dt}\|F_t^n\|_s \leqslant\ & C_s^{(6)} (1 + \varepsilon \|F_0^n\|_0) \|F_0^n\|_2 \|F_t^n\|_s \\ & -C_s^{(4)} \|F_0^n\|_0 \left(\|F_0^n\|_2\right)^{-\gamma/(s-2)} \left(\|F_t^n\|_s\right)^{1+\gamma/(s-2)} \qquad \forall t > 0 \end{aligned}$$

which implies that (see Wennberg[31], Bobylev[2], or Lu[19])

$$M_s(F_t^n) < \|F_t^n\|_s \leqslant C^{(s-2)/\gamma} (1 - e^{-at})^{-(s-2)/\gamma} \qquad \forall t > 0$$

where

$$\begin{aligned} a &= \frac{\gamma}{s-2} C_s^{(6)} (1 + \varepsilon \|F_0^n\|_0) \|F_0^n\|_2, \\ C &= \frac{C_s^{(6)} (1 + \varepsilon \|F_0^n\|_0)(\|F_0^n\|_2)^{1+\gamma/(s-2)}}{C_s^{(4)} \|F_0^n\|_0}. \end{aligned}$$

Since

$$(1 - e^{-at})^{-(s-2)/\gamma} \leqslant (1 + 1/a)^{(s-2)/\gamma} (1 + 1/t)^{(s-2)/\gamma}$$

and $\|F_0^n\|_0 = M_0(F_0^n)$, $\|F_0^n\|_2 = M_0(F_0^n) + M_2(F_0^n)$, this gives the estimate (5.5) for $s \geqslant 4$. ∎

6. EQUILIBRIUM STATES AND BEC

In this section we will give a rigorous derivation of the Bose-Einstein condensation for equilibrium states using the weak form (2.12) of Eq. (BBE).

Let F be a finite positive Borel measure on $\mathbf{R}_+$ with finite moments up to order 2 and $\int_{\mathbf{R}_+} dF(r) > 0$. Let F_t be an isotropic distributional solution to Eq. (BBE) with initial datum $F_t|_{t=0} = F$ and suppose that F_t conserves the mass and energy. According to Chapman and Cowling[(9)], Chap. 2 and Truesdell and Muncaster[(27)], pp. 43–44, the kinetic temperature of the corresponding particle system is defined by

$$\overline{T} = \frac{m}{2} \cdot \frac{E}{N} \cdot \frac{2}{3k_B} \tag{6.1}$$

where

$$N = \int_{\mathbf{R}_+} dF(r), \quad E = \int_{\mathbf{R}_+} r^2 dF(r), \tag{6.2}$$

N is the total number of particles per unit space volume, $\frac{1}{2}mE$ is the total kinetic energy per unit space volume, m is the mass of a particle and k_B is the Boltzmann constant. Note that here the "kinetic temperature" is used to distinguish the usual "temperature" because it is different from the classical cases that in the quantum cases there are no linear relation between the kinetic temperature and the temperature (see also (6.13) for equilibrium states).[2]

A distributional solution F_t to the Eq. (BBE) in the weak form (2.12) is called an equilibrium solution (or an equilibrium state) if F_t does not depend on t. This is equivalent to that

$$F_t = F \quad \forall t \geqslant 0 \qquad \text{and} \qquad \langle Q_B(F), \varphi\rangle = 0 \qquad \forall \varphi \in C_b^2(\mathbf{R}_+).$$

The Bose-Einstein distribution with zero mean-velocity is given by (see Chapman and Cowling [**9**, Chap. 17] and recall that $1/\varepsilon = \frac{m^3 \mathrm{g}}{h^3}$)

$$B_{\alpha,\beta}(|v|) = \frac{m^3 \mathrm{g}}{h^3} \cdot \frac{1}{e^{\alpha + \beta m|v|^2/2} - 1}, \quad \beta = \frac{1}{k_B T} \tag{6.3}$$

[2]According to this distinction, the temperature used in refs. 20–22, should be understood as the kinetic temperature.

where $\alpha \geqslant 0$ is a constant and $T > 0$ is the temperature. Let $\delta_0(v)$ be the standard Dirac δ-function on $\mathbf{R}^3$ concentrating at $v = 0$. The δ-function $\delta_0(v)$ determines the Dirac measure μ_0 on $\mathbf{R}_+$ by

$$\int_{\mathbf{R}_+} \psi(r) d\mu_0(r) = \int_{\mathbf{R}^3} \psi(|v|)\delta_0(v) dv = \psi(0), \quad \psi \in C(\mathbf{R}_+). \tag{6.4}$$

Now we consider the following type of distributions consisting of Bose-Einstein distributions and Dirac measures:

$$dF(r) = 4\pi r^2 B_{\alpha,\beta}(r) dr + N_0 \, d\mu_0(r) \tag{6.5}$$

where $N_0 \geqslant 0$ is a constant.

It should be noted that if F in (6.5) is an equilibrium solution to Eq. (BBE), then the T given through $\beta = 1/k_B T$ is the temperature of the corresponding particle system, but if F is not an equilibrium solution, T should be only regarded as a parameter for defining F.

In contract to traditional result on BEC, we will use the critical temperature T_c,

$$T_c = \frac{h^2}{2\pi m k_B} \left[\frac{N}{\zeta(3/2) \mathrm{g}} \right]^{2/3} \tag{6.6}$$

which is derived from the model of ideal Bose gases (see Landau and Lifshitz[17], pp. 180–181; p. 36, or Pathria[23], p. 180 for $\mathrm{g} = 1$ (spinless)), where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(s > 1)$ is the Riemann-Zeta function.

With the above notations we can state our main result of this section as follows:

Theorem 5. Let F be the positive Borel measure (on $\mathbf{R}_+$) given by (6.5) with $T > 0, \alpha \geqslant 0$ and $N_0 \geqslant 0$.

Then

(I) F is an equilibrium solution of Eq. (BBE) in the weak form (2.12) if and only if $\alpha = 0$ or $N_0 = 0$.

(II) Let $\overline{T}$, N be given in (6.1)–(6.2) for the present F and let T_c be given in (6.6). Let

$$\overline{T}_c := \frac{\zeta(5/2)}{\zeta(3/2)} T_c \approx 0.5134 \, T_c.$$

Then

$$\frac{N_0}{N} \geqslant 1 - \left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5} \geqslant 1 - \left(\frac{T}{T_c}\right)^{3/2}. \tag{6.7}$$

(III) (Exact ratio of BEC)

$$\frac{N_0}{N} = 1 - \left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5} \iff \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \tag{6.8}$$

$\iff F$ is an equilibrium solution with $\alpha = 0$.

For each of the cases in (III), it necessarily holds the low temperature condition $T \leqslant T_c$ (or equivalently $\overline{T} \leqslant \overline{T}_c$).

Remark 3. (1) Escobedo, Mischler and Valle in ref. 13 considered the conditional maximum entropy (here the entropy is the Boltzmann H-functional (times -1) corresponding to Eq. (BBE)) and proved that the entropy takes its conditional maximum value at a positive measure F if and only if F takes the form (6.5) with $\alpha = 0$ or $N_0 = 0$. Thus the above Theorem 5 shows that every entropy maximizer is an equilibrium solution to Eq. (BBE) in the weak form (2.12). It is physically believed that every equilibrium solution to Eq. (BBE) is also an entropy maximizer and therefore it takes the form (6.5) with $\alpha = 0$ or $N_0 = 0$, but the proof will be very difficult because the equation is now taken in the weak form (in order to include the hard sphere model) and a solution may have a singular part which is not merely a single Dirac measure (see ref. 13 for discussions).

(2) At low temperatures (e.g. $T < T_c$ or $\overline{T} < \overline{T}_c$), the condition $\alpha = 0$ is equivalent to that the chemical potential is zero. This is consistent with the traditional result for ideal Bose gases (see for instance Huang [(16)], pp. 262–265, Landau and Lifshitz [(17)], pp. 180–181]). The exact ratio in (6.8) for Bose-Einstein condensation is essentially the same to those derived in text books for equilibrium states (see e.g. refs. 16,17,23 where a ratio for BEC for an ideal Bose gas is given by (with the same meaning of N and N_0)

$$\frac{N_0}{N} \approx 1 - \left(\frac{T}{T_c}\right)^{3/2}, \qquad T < T_c.$$

Here, as well known, the approximation "$\approx$" is due to the discreteness of the pure quantum model. In view of a discussion by Cercignani[(8)] for continuous and discrete models, there may be some differences about temperature, entropy, etc. between the BBE model (which is in fact a semi-classical one) and the pure quantum model (for idea Bose gases), but here we will not make further comparisons.

Proof of Theorem 5. We first prove Part (II) and Part (III) (using Part (I)). The proof of Part (I) will be given later. Let F be given by (6.5) and let

$$N_e = \int_{\mathbf{R}_+} 4\pi r^2 B_{\alpha,\beta}(r)dr. \tag{6.9}$$

Following the calculation in ref. 20, Section 5 and noting that the Dirac measure $N_0\mu_0$ has no contribution to the energy E, i.e., $E = \int_{\mathbf{R}_+} 4\pi r^4 B_{\alpha,\beta}(r)dr$, we have

$$\frac{E}{(N_e)^{5/3}} = \varepsilon^{2/3} \cdot \frac{3}{2\pi} \cdot \mathrm{R}(e^{-\alpha}) \tag{6.10}$$

where

$$\mathrm{R}(t) = \frac{\mathrm{P}_{5/2}(t)}{[\mathrm{P}_{3/2}(t)]^{5/3}}, \quad \mathrm{P}_s(t) = \sum_{n=1}^{\infty} n^{-s} t^n, \quad t \in (0,1], \quad s > 1$$

and $\mathrm{R}(t)$ is strictly decreasing on $t \in (0,1]$. Note that $\mathrm{P}_s(1) = \zeta(s)$. Since

$$\mathrm{R}(1) = \frac{\mathrm{P}_{5/2}(1)}{[\mathrm{P}_{3/2}(1)]^{5/3}} = \frac{\zeta(5/2)}{[\zeta(3/2)]^{5/3}},$$

it follows from definitions of $\overline{T}$, $\overline{T}_c$ and ε and using (6.10) that

$$\frac{\overline{T}}{\overline{T}_c} = \left(\frac{N_e}{N}\right)^{5/3} \frac{\mathrm{R}(e^{-\alpha})}{\mathrm{R}(1)} \quad \text{i.e.} \quad \frac{N_e}{N} = \left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5} \left(\frac{\mathrm{R}(1)}{\mathrm{R}(e^{-\alpha})}\right)^{3/5}. \tag{6.11}$$

On the other hand, from (6.9) we compute

$$N_e = \mathrm{g}\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \mathrm{P}_{3/2}(e^{-\alpha})$$

so that

$$\frac{N_e}{N}=\left(\frac{T}{T_c}\right)^{3/2}\frac{\mathrm{P}_{3/2}(e^{-\alpha})}{\mathrm{P}_{3/2}(1)}. \tag{6.12}$$

Comparing (6.11) with (6.12) we obtain by a simple calculation that

$$\left(\frac{T}{T_c}\right)^{3/2}=\left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5}\left(\frac{\mathrm{P}_{5/2}(1)}{\mathrm{P}_{5/2}(e^{-\alpha})}\right)^{3/5}. \tag{6.13}$$

Since $\mathrm{R}(e^{-\alpha})\geqslant \mathrm{R}(1)$ and $\mathrm{P}_{5/2}(e^{-\alpha})\leqslant \mathrm{P}_{5/3}(1)$ for $\alpha\geqslant 0$, it follows from (6.11) and (6.13) that

$$\frac{N_e}{N}\leqslant\left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5}\leqslant\left(\frac{T}{T_c}\right)^{3/2}$$

which implies (6.7) because $N_e=N-N_0$. This proves Part (II). To prove Part (III) we note that the equivalency in (6.8) can be written

$$\frac{N_e}{N}=\left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5}\iff\frac{N_e}{N}=\left(\frac{T}{T_c}\right)^{3/2}. \tag{6.14}$$

Since $\mathrm{R}(t)$ and $\mathrm{P}_{3/2}(t)$ are both strictly monotone on $t\in(0,1]$, comparing (6.11) and (6.12) with the two equalities in (6.14) and noting that $\overline{T}>0, T>0$, one sees that each of the two equalities in (6.14) is equivalent to $\alpha=0$. While, by Part (I), the condition $\alpha=0$ implies that F is an equilibrium solution to Eq. (BBE) in the weak form (2.12). Conversely, if F is an equilibrium solution with $\alpha=0$, then, as shown above, the two equalities in (6.14) hold simultaneously. This proves Part(III). The necessity of the low temperature condition $T\leqslant T_c$ (or $\overline{T}\leqslant\overline{T}_c$) for Part (III) is obvious because $N_0\geqslant 0$.

Now we are going to prove Part (I). By replacing F with εF and $B(v-v_*,\omega)$ with $\frac{1}{\varepsilon^2}B(v-v_*,\omega)$ we may assume that $\varepsilon=1$. We first prove that the Bose-Einstein distribution in (6.3) and the Dirac measure $N_0\mu_0$ in (6.4) are both equilibria of Eq. (BBE) in the weak form (2.12). Since $J_B[\varphi](r,r_*)$, $K_B[\varphi](r,r',r'_*)$ are continuous on $(r,r_*)\in\mathbf{R}^2_+$ and on $(r,r',r'_*)\in\mathbf{R}^3_+$ respectively, and satisfies $J_B[\varphi](0,0)=0$ and

$$K_B[\varphi](0,0,r'_*)=K_B[\varphi](r,r',0)=0\qquad\forall\, r,r',r'_*\geqslant 0 \tag{6.15}$$

it follows from definition of $\langle Q_B(F), \varphi\rangle$ that the Dirac measure $\mu = N_0\mu_0$ is an equilibrium, i.e., $\langle Q_B(\mu), \varphi\rangle = 0$ for all $\varphi \in C_b^2(\mathbf{R}_+)$. For the Bose-Einstein distribution $B_{\alpha,\beta}(|v|)$ with $\varepsilon = 1$, it is well known that it satisfies the following equation

$$f' f'_*(1 + f + f_*) = f f_*(1 + f' + f'_*). \tag{6.16}$$

Let $dG(r) = 4\pi r^2 B_{\alpha,\beta}(r)dr$ and let $B_n(v - v_*, \omega)$ be the cutoff kernels (2.1). Since the function $f(|v|) := B_{\alpha,\beta}(|v|)$ belongs to $L^1_2(\mathbf{R}^3)$, it follows from our derivation in Section 2 for the weak form of Eq. (BBE) and (6.16) that for all $\varphi \in C_b^2(\mathbf{R}_+)$

$$\begin{aligned}\langle Q_{B_n}(G), \varphi\rangle = &\iint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B_n(v - v_*, \omega)\varphi(|v|^2) \\ &\times \left[f' f'_*(1 + f + f_*) - f f_*(1 + f' + f'_*)\right] d\omega dv dv_* = 0.\end{aligned}$$

Thus by Lemma 3, Lemma 4 (with $\mu = \mu_n = G, k = 2, 3$ and $s = 2$) we obtain

$$\langle Q_B(G), \varphi\rangle = \lim_{n\to\infty} \langle Q_{B_n}(G), \varphi\rangle = 0 \qquad \forall \varphi \in C_b^2(\mathbf{R}_+).$$

Next we prove that the measure $dF(r) = dG(r) + d\mu(r)$ with $dG(r) = 4\pi r^2 f(r)dr$, $f(r) = B_{\alpha,\beta}(r)$ (for $\varepsilon = 1$) and $\mu = N_0\mu_0$ (for $N_0 > 0$) is an equilibrium solution to Eq. (BBE) in the weak form (2.12) if and only if $\alpha = 0$. First of all we have the following decomposition

$$\langle Q_B(F), \varphi\rangle = \langle Q_B(G), \varphi\rangle + \langle Q_B(\mu), \varphi\rangle + \langle R_B(F), \varphi\rangle + \langle S_B(F), \varphi\rangle$$

where as we have shown that $\langle Q_B(G), \varphi\rangle = \langle Q_B(\mu), \varphi\rangle = 0$ and

$$\begin{aligned}&\langle R_B(F), \varphi\rangle \\ &= \iint_{\mathbf{R}^2_+} J_B[\varphi](r, r_*) dG(r) d\mu(r_*) + \iint_{\mathbf{R}^2_+} J_B[\varphi](r, r_*) dG(r_*) d\mu(r),\end{aligned}$$

$$\begin{aligned}\langle S_B(F), \varphi\rangle = &\iiint_{\mathbf{R}^3_+} K_B[\varphi](r, r', r'_*) dG(r) dG(r') d\mu(r'_*) \\ &+ \iiint_{\mathbf{R}^3_+} K_B[\varphi](r, r', r'_*) dG(r) dG(r'_*) d\mu(r') \\ &+ \iiint_{\mathbf{R}^3_+} K_B[\varphi](r, r', r'_*) dG(r') dG(r'_*) d\mu(r) \\ &+ \iiint_{\mathbf{R}^3_+} K_B[\varphi](r, r', r'_*) dG(r) d\mu(r') d\mu(r'_*)\end{aligned}$$

$$+\iiint_{\mathbf{R}_+^3} K_B[\varphi](r,r',r'_*)dG(r')d\mu(r)d\mu(r'_*)$$
$$+\iiint_{\mathbf{R}_+^3} K_B[\varphi](r,r',r'_*)dG(r'_*)d\mu(r)d\mu(r').$$

Since the measure $\mu=N_0\mu_0$ concentrates at $r=0$, it follows from (6.15) and $dG(r)=4\pi r^2 f(r)dr$ that

$$\begin{aligned}
&\langle R_B(F),\varphi\rangle\\
&=N_0 4\pi\int_{\mathbf{R}_+} J_B[\varphi](r,0)r^2 f(r)dr+N_0 4\pi\int_{\mathbf{R}_+} J_B[\varphi](0,r_*)r_*^2 f(r_*)dr_*,\\
&\langle S_B(F),\varphi\rangle=N_0(4\pi)^2\iint_{\mathbf{R}_+^2} K_B[\varphi](r,0,r'_*)r^2 f(r)r_*'^2 f(r'_*)drdr'_*\\
&\qquad +N_0(4\pi)^2\iint_{\mathbf{R}_+^2} K_B[\varphi](0,r',r'_*)r'^2 f(r')r_*'^2 f(r'_*)dr'dr'_*.
\end{aligned}$$

Making change of variables: let $r=\rho\cos\theta, r'_*=\rho$ for $K_B[\varphi](r,0,r'_*)$ (this is because $K_B[\varphi](r,0,r'_*)=K_B[\varphi](r,0,r'_*)\mathbf{1}_{\{r'_*>r\}}$), and let $r'=\rho\cos\theta, r'_*=\rho\sin\theta$ for $K_B[\varphi](0,r',r'_*)$. Then

$$\begin{aligned}
&\frac{1}{N_0(4\pi)^2}\langle Q_B(F),\varphi\rangle=\frac{1}{4\pi}\int_0^\infty [J_B[\varphi](\rho,0)+J_B[\varphi](\rho,0)]\,\rho^2 f(\rho)d\rho\\
&+\int_0^\infty \rho^5\int_0^{\pi/2} K_B[\varphi](\rho\cos\theta,0,\rho)\cos^2\theta\sin\theta f(\rho\cos\theta)f(\rho)d\theta d\rho\\
&+\int_0^\infty \rho^5\int_0^{\pi/2} K_B[\varphi](0,\rho\cos\theta,\rho\sin\theta)\cos^2\theta\sin^2\theta\\
&\times f(\rho\cos\theta)f(\rho\sin\theta)d\theta d\rho.
\end{aligned}$$

Calculation using (2.11) gives

$$J_B[\varphi](\rho,0)+J_B[\varphi](0,\rho)=4\pi\int_0^{\pi/2} B(\rho,\cos\theta)\sin\theta\,\Delta\varphi(\rho\cos\theta,0,\rho)d\theta$$

where we have used the definition of $\Delta\varphi(r,r',r'_*)$, which gives

$$\Delta\varphi(\rho\cos\theta,0,\rho)=\varphi(\rho^2\cos^2\theta)+\varphi(\rho^2\sin^2\theta)-\varphi(0)-\varphi(\rho^2).$$

So

$$\frac{1}{4\pi}\int_0^\infty [J_B[\varphi](\rho,0)+J_B[\varphi](0,\rho)]\,\rho^2 f(\rho)d\rho$$

$$= \int_0^\infty \rho^2 f(\rho) \int_0^{\pi/2} B(\rho, \cos\theta) \sin\theta \Delta\varphi(\rho\cos\theta, 0, \rho) d\theta d\rho.$$

Also by definition of $K_B[\varphi](r, r', r'_*)$ (or using (2.10)) and observing that $\Delta\varphi(0, \rho\cos\theta, \rho\sin\theta) = -\Delta\varphi(\rho\cos\theta, 0, \rho)$, we compute

$$\begin{aligned} &K_B[\varphi](\rho\cos\theta, 0, \rho)\cos^2\theta\sin\theta \\ &= \rho^{-3}\left[B(\rho, \cos\theta)\sin\theta + B(\rho, \sin\theta)\cos\theta\right]\Delta\varphi(\rho\cos\theta, 0, \rho), \end{aligned}$$

$$\begin{aligned} &K_B[\varphi](0, \rho\cos\theta, \rho\sin\theta)\cos^2\theta\sin^2\theta \\ &= -\frac{1}{2}\rho^{-3}\left[B(\rho, \cos\theta)\sin\theta + B(\rho, \sin\theta)\cos\theta\right]\Delta\varphi(\rho\cos\theta, 0, \rho). \end{aligned}$$

By symmetry $\Delta\varphi(\rho\sin\theta, 0, \rho) = \Delta\varphi(\rho\cos\theta, 0, \rho)$, this gives

$$\begin{aligned} &\int_0^\infty \rho^5 \int_0^{\pi/2} K_B[\varphi](\rho\cos\theta, 0, \rho)\cos^2\theta\sin\theta f(\rho\cos\theta) f(\rho) d\theta d\rho \\ &= \int_0^\infty \rho^2 \int_0^{\pi/2} B(\rho, \cos\theta)\sin\theta\Delta\varphi(\rho\cos\theta, 0, \rho) \\ &\quad \times f(\rho)\left[f(\rho\cos\theta) + f(\rho\sin\theta)\right] d\theta d\rho, \end{aligned}$$

$$\begin{aligned} &\int_0^\infty \rho^5 \int_0^{\pi/2} K_B[\varphi](0, \rho\cos\theta, \rho\sin\theta)\cos^2\theta\sin^2\theta f(\rho\cos\theta) f(\rho\sin\theta) d\theta \\ &= -\int_0^\infty \rho^2 \int_0^{\pi/2} B(\rho, \cos\theta)\sin\theta\Delta\varphi(\rho\cos\theta, 0, \rho) f(\rho\cos\theta) f(\rho\sin\theta) d\theta d\rho. \end{aligned}$$

Summarizing above gives

$$\begin{aligned} \frac{1}{N_0(4\pi)^2}\langle Q_B(F), \varphi\rangle &= \int_0^\infty \rho^2 \int_0^{\pi/2} B(\rho, \cos\theta)\sin\theta\Delta\varphi(\rho\cos\theta, 0, \rho) \\ &\times \left\{ f(\rho)\left[1 + f(\rho\cos\theta) + f(\rho\sin\theta)\right] - f(\rho\cos\theta) f(\rho\sin\theta) \right\} d\theta d\rho. \end{aligned} \tag{6.17}$$

It should be noted that in the above derivation there are no integrability problems even for the "worst" case $\alpha = 0$ because $|\Delta\varphi(\rho\cos\theta, 0, \rho)| \leqslant 3\|D^2\varphi\|_{L^\infty}\rho^4\cos^2\theta\sin^2\theta$ (see Lemma 1).

The rest of proof is based on the following observation for the Bose-Einstein distribution $f(r) = B_{\alpha,\beta}(r) = (e^{\alpha+\beta m r^2/2} - 1)^{-1}$ (with $\varepsilon = 1$): If $\alpha = 0$, then

$$f(r)f(r_*) = f(\sqrt{r^2 + r_*^2})\,[1 + f(r) + f(r_*)]\,.$$

By (6.17), this immediately implies that if $\alpha = 0$ then $\langle Q_B(F), \varphi\rangle = 0$ for all $\varphi \in C_b^2(\mathbf{R}_+)$, i.e., F is an equilibrium solution to Eq. (BBE). For general case ($\alpha \geqslant 0$), we have

$$f(r)f(r_*) = f(\sqrt{r^2 + r_*^2})\,[1 + f(r) + f(r_*)] - R(r, r_*) \tag{6.18}$$

where

$$R(r, r_*) = \frac{(e^{\alpha} - 1) f(r) f(r_*)}{1 - e^{-\alpha - \beta m (r^2 + r_*^2)/2}}. \tag{6.19}$$

Now from (6.17) and (6.18) we obtain

$$\frac{1}{N_0 (4\pi)^2} \langle Q_B(F), \varphi\rangle = \int_0^{\infty} \rho^2 \int_0^{\pi/2} B(\rho, \cos\theta) \sin\theta\, \Delta\varphi(\rho\cos\theta, 0, \rho) R(\rho\cos\theta, \rho\sin\theta) d\theta d\rho. \tag{6.20}$$

Suppose that $\alpha > 0$. Then (6.19) implies $R(r, r*) > 0$ for all $r, r* \geqslant 0$. If we choose the test function as $\varphi(r) = -e^{-r}$, then $\Delta\varphi(\rho\cos\theta, 0, \rho) > 0$ for all $(\rho, \theta) \in (0, \infty) \times (0, \pi/2)$. Since $\int_0^1 B(V, \tau) d\tau > 0$ for all $V > 0$, it follows from (6.20) that $\langle Q_B(F), \varphi\rangle > 0$ which means that F is not an equilibrium solution to Eq. (BBE) in the weak form (2.12). This proves Part(I).

ACKNOWLEDGMENTS

I am very grateful to the referees for their valuable suggestions on the revision of the paper. This work was supported by the National Natural Science Foundation of China-Grant No. 10171053.

REFERENCES

1. L. Arkeryd, Intermolecular forces of infinite range and the Boltzmann equation, *Arch. Rational Mech. Anal.* **77**:11–21 (1981).
2. A. V. Bobylev, Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems, *J. Stat. Phys.* **88**:1183–1214 (1997).
3. T. Carleman, *Problèmes mathématiques dans la théorie cinétiquedes gaz* (Almqvist and Wiksell, Uppsala, 1957).

4. E. A. Carlen, M. C. Carvalho, and E. Gabetta, Central limit theorem for Maxwellian molecules and truncation of the Wild expansion, *Comm. Pure Appl. Math.* **53**:370–397 (2000).
5. E. A. Carlen and X. G. Lu, Fast and Slow Convergence to Equilibrium for Maxwellian Molecules via Wild Sums, *J. Statist. Phys.* **112**:59–134 (2003).
6. C. Cercignani, *The Boltzmann equation and its applications* (Applied Mathematical Sciences, **67**, Springer-Verlag, New York, 1988).
7. C. Cercignani, R. Illner, and M. Pulvirenti, *The mathematical theory of dilute gases* (Applied Mathematical Sciences, **106** Springer-Verlag, New York, 1994).
8. C. Cercignani, Temperature, entropy, and kinetic theory, *J. Stat. Phys.* **87**:1097–1109 (1997).
9. T. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Third Edition (Cambridge University Press, 1970).
10. J. Dolbeault, Kinetic models and quantum effects: a modified Boltzmann equation for Fermi-Dirac particles, *Arch. Rational Mech. Anal.* **127**:101–131 (1994).
11. M. Escobedo and S. Mischler, On a quantum Boltzmann equation for a gas of photons, *J. Math. Pures Appl.* **80**:471–515 (2001).
12. M. Escobedo, Boltzmann equation for quantum particles, and Fokker Planck approximation, *Elliptic and parabolic problems* (Rolduc/Gaeta, 2001), 75–87, World Sci. Publishing, River Edge, NJ, 2002.
13. M. Escobedo, S. Mischler, and M. A. Valle, Homogeneous Boltzmann equation in quantum relativistic kinetic theory, *Electronic Journal of Differential Equations*, (Monograph, 4 Southwest Texas State University, San Marcos, TX, 2003).
14. E. Gabetta, G. Toscani, and B. Wennberg, Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation, *J. Statist. Phys.* **81**:901–934 (1995).
15. T. Goudon, On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions, *J. Statist. Phys.* **89**:751–776 (1997).
16. K. Huang, *Statistical Mechanics* (John Wiley, and Sons, Inc., New York, London, 1963).
17. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Third Edition, Part 1 (Pergamon Press, 1980).
18. P. L. Lions, Compactness in Boltzmann's equation via Fourier integral operators and applications, III, *J. Math. Kyoto Univ.* **34**:539–584 (1994).
19. X. G. Lu, Conservation of energy, entropy identity, and local stability for the spatially homogeneous Boltzmann equation, *J. Statist. Phys.* **96**:765–796 (1999).
20. X. G. Lu, A modified Boltzmann equation for Bose-Einstein particles: isotropic solutions and long-time behavior, *J. Statist. Phys.* **98**:1335–1394 (2000).
21. X. G. Lu, On spatially homogeneous solutions of a modified Boltzmann equation for Fermi-Dirac particles, *J. Statist. Phys.* **105**:353–388 (2001).
22. X. G. Lu and B. Wennberg, On stability and strong convergence for the spatially homogeneous Boltzmann equation for Fermi-Dirac particles, *Arch. Rational Mech. Anal.* **168**:1–34 (2003).
23. R. K. Pathria, *Statistical Mechanics* (Pergamon Press, 1972).
24. D. V. Semikov and I. I. Tkachev, Kinetics of Bose condensation, *Phys. Rev. Lett.* **74**:3093–3097 (1995).
25. D.V. Semikov and I. I. Tkachev, Condensation of Bose in the kinetic regime, *Phys. Rev. D* **55**:489–502 (1997).
26. G. Toscani and C. Villani, Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas, *J. Statist. Phys.* **94**:619–637 (1999).

27. C. Truesdell and R. G. Muncaster, *Fundamentals Maxwell's Kinetic Theory of a Simple Monoatomic Gas* (Academic Press, New York, 1980).
28. E. A. Uehling and G. E. Uhlenbeck, Transport phenomena in Einstein-Bose and Fermi-Dirac gases, I, *Phys. Rev.* **43**:552–561 (1933).
29. C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, *Arch. Rational Mech. Anal.* **143**:273–307 (1998).
30. C. Villani, A review of mathematical topics in collisional kinetic theory, *Handbook of mathematical fluid dynamics*, Vol.I, pp.71–305, (North-Holland, Amsterdam, 2002).
31. B. Wennberg, Entropy dissipation and moment production for the Boltzmann equation, *J. Statist. Phys.* **86**:1053–1066 (1997).